PART I: CHAPTER FOUR
CARRIER-WAVE MODULATION

Introduction

In Chapter 1, we introduced the concept of generalized harmonic analysis. That mathematical tool allowed us to apply Fourier series to the analysis of non-linear processes. In this chapter, we will look at instances where the generalized harmonics have arguments consisting of a simple carrier and a modulation term. We will illustrate the principle of carrier modulation by two examples. First, we will begin with a simple cosine carrier and then move to the more complicated square wave carrier. In both cases, we will see that the resulting complex amplitude transmittances can be expressed in terms of a superposition of images. For most applications, we impress one desired image on the carrier. This modulation image we term the true image. As a consequence of the modulation, other images are generated. These undesired images we term false images.

Cosine-Carrier Modulation

We will look at the encoding of complex information on a real cosine carrier. We begin by Hermitian symmetrizing the object about the origin.

\[ u(x) = 0 \quad \text{for } |x| > \frac{\Delta x}{2} \]

\[ u_h(x) = \frac{1}{2} \left[ u(x + \frac{\Delta x}{2}) + u^*(x - \frac{\Delta x}{2}) \right] \]

We then Fourier transform and find that the symmetrization has led to displacements which transform into linear phase factors.

\[ U_h(\xi) = \frac{1}{2} \left[ U(\xi) e^{+ \frac{2\pi i \Delta x \xi}{2}} + U^*(\xi) e^{- \frac{2\pi i \Delta x \xi}{2}} \right] \]

Converting the object Fourier transform into polar form,

\[ U(\xi) = A(\xi) e^{i\phi(\xi)} \]

We can write the total transform as a modulation in amplitude and phase of a cosine carrier.
\[
U_h(\xi) = \frac{1}{2} \left( A(\xi)e^{+2\pi i \left( \frac{\Delta x \xi}{2} + \frac{\phi(\xi)}{2\pi} \right)} + A(\xi)e^{-2\pi i \left( \frac{\Delta x \xi}{2} + \frac{\phi(\xi)}{2\pi} \right)} \right) = A(\xi)\cos \left\{ 2\pi \left( \frac{\Delta x \xi}{2} + \frac{\phi(\xi)}{2\pi} \right) \right\}
\]

We also see that the end result is equivalent to simply taking the original object a distance \( \frac{\Delta x}{2} \) off axis, Fourier transforming and then taking the real part of the Fourier transform. This procedure has an advantage in simplicity over doing the actual symmetrization.

**Square-Wave Carrier Modulation**

The basic formula is the Fourier series for a square-wave of duty cycle \( a \) and unit period,

\[
Sq(a, x) = \sum_{n=-\infty}^{\infty} a \sin(na) e^{2\pi inx}
\]

In order to pave the way for some meaningful substitutions, instead of \( x \), we will use the variable \( p \) so that

\[
Sq(a, p) = \sum_{n=-\infty}^{\infty} a \sin(na) e^{2\pi inp}
\]

In pulse-code modulation (PCM), a square-wave replaces the cosine as the carrier in the calculations. Impressing the modulation on the carrier is done by generalized harmonic analysis, i.e., first expanding the square-wave as a Fourier series and then substituting more complicated functions for the simple arguments. The more complicated arguments are

\[
a(\xi) = \frac{1}{\pi} \arcsin[A(\xi)]; \quad p(\xi) = \left[ \frac{\Delta x \xi + \phi(\xi)}{2\pi} \right]
\]

The leading factor of \( \frac{1}{\pi} \) is inserted so that when \( A(\xi) = 1 \) the duty cycle is \( a(\xi) = \frac{1}{\pi} \arcsin[1] = \frac{\pi}{2} = \frac{1}{2} \).

The substitutions result in

\[
V(\xi) = Sq \left\{ \frac{1}{\pi} \arcsin[A(\xi)]; \Delta x \xi + \frac{\phi(\xi)}{2\pi} \right\}
\]

\[
= \sum_{m=-\infty}^{\infty} \frac{1}{\pi} \arcsin[A(\xi)] \sin \left\{ m \frac{1}{\pi} \arcsin[A(\xi)] \right\} e^{2\pi im \left[ \frac{\Delta x \xi + \phi(\xi)}{2\pi} \right]}
\]

which simplifies to

\[
V(\xi) = \sum_{m=-\infty}^{\infty} \frac{1}{\pi} \arcsin[A(\xi)] \sin \left\{ \frac{m}{\pi} \arcsin[A(\xi)] \right\} e^{im\phi(\xi)} e^{2\pi im \Delta x \xi}
\]

With the definition

\[
U_m(\xi) = \frac{1}{\pi} \arcsin[A(\xi)] \sin \left\{ \frac{m}{\pi} \arcsin[A(\xi)] \right\} e^{im\phi(\xi)}
\]

\[
= \frac{1}{\pi} \arcsin[A(\xi)] \sin \left\{ \frac{m}{\pi} \arcsin[A(\xi)] \right\} e^{im\phi(\xi)} = \frac{1}{m \pi} \sin \left\{ m \arcsin[A(\xi)] \right\} e^{im\phi(\xi)}
\]

we have that
\[ V(\xi) = \sum_{m=-\infty}^{\infty} U_m(\xi) e^{2\pi im\Delta x \xi} \]

The important consideration is the reconstruction at the first diffraction order, i.e., the structure of the \( m = 1 \) term. We see that its structure is really quite simple,

\[ U_1(\xi) e^{2\pi i\Delta x \xi} = \frac{e^{2\pi i\Delta x \xi}}{\pi} \sin \{ \arcsin[A(\xi)] \} e^{im\phi(\xi)} = \frac{e^{2\pi i\Delta x \xi}}{\pi} A(\xi) e^{i\phi(\xi)} = \frac{e^{2\pi i\Delta x \xi}}{\pi} U(\xi) \]

The inverse Fourier transform of this term is

\[ \frac{1}{\pi} u(x + \Delta x) \]

We have purposefully made the displacement for this case twice as large as the displacement we used for the cosine modulation case. There is a modulation image due to the \( m=0 \) term that is located on the optical axis. It is to move away from this disturbing influence that the displacement is increased.

**The Modulation Image Superposition**

Although the appearance is of a very neat air-tight calculation, there is an Achilles heel; the higher orders are not necessarily of bounded support even when the original object is. The higher orders may leak over into the desired reconstruction. The second consideration is that pulse code modulation calculation done here is in continuous mathematics. When the hologram transmittance is sampled, this sampling will lead to a periodic replication of the entire reconstruction plane thereby bringing higher diffraction orders into registration with the desired reconstruction region.

It is convenient here to introduce the term "modulation image". The modulation image is the Fourier transform of a modulated amplitude associated with a carrier for a given order. The total image is a sum of the modulation images. The modulation images are spatially translated due to their association with the various carriers but may overlap. We will term the modulation image, which is the desired reconstruction, to be the true image. Other images in this super position we will call "false images". As an example, in the square wave decomposition above, the modulation image associated with the \( m = 1 \) carrier, is the true image. The images associated with all other carriers, are false images.