Introduction

In this section we deal with the five sources of image distortion: phase quantization noise, phase clipping, phase matching, the linear phase error in detour-phase holograms, and the constant modulus error in kinoforms. Each of these distortions is conveniently described as a Fourier-domain point nonlinearity.

Complex Amplitude Distortions

- Degradation of the complex amplitude vs irradiance
- Phase nonlinearities
- Display vs diffractive optical elements

Sampling and Point-Oriented CGH’s

Until now we have considered continuous transmittances in our discussion of binarization. We have, however, ignored sampling. At this point we include sampling in our consideration. It is important to note that for this type of hologram, the order of binarization and sampling is interchangeable. Sampling can easily be described by multiplication with a two-dimensional comb with the hologram transmittance.

\[ T_{CGH} \rightarrow T_{CGH}comb(\Delta s x', \Delta s y')comb(\Delta s x, \Delta s y) \]

This multiplication in the Fourier domain translates to a convolution in the spatial domain. This convolution is a replication of the entire reconstruction plane which when expressed in terms of the false images, becomes

\[ t_{CGH} \rightarrow \frac{1}{\Delta x_s} \frac{1}{\Delta y_s} t_{CGH} * comb\left(\frac{x}{\Delta x_s}\right)comb\left(\frac{y}{\Delta y_s}\right) \]

This equation simplifies for particular choices of the replication distance which is determined by the sampling interval. Let’s take two examples

First the sampling interval being equal to the object extent

\[ \Delta x_s = \Delta x \quad \Delta y_s = \Delta y \]
For this example we have a false-image decomposition

\[ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{mn} u_{mn} (x - m\Delta x, y - n\Delta y) \]

The strengths of the images in the desired reconstruction region are given by the coefficients

\[ C_{mn} = \frac{1}{2} \sin \left( \frac{m}{2} \right) \]

We see that the strength of the false images dies off as a sinc-function.

For the second example, we choose the replication distance to be twice the desired object extent. The false image decomposition is

\[ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{mn} u_{mn} (x - 2m\Delta x, y - 2n\Delta y) \]

the strengths of the images in the desired reconstruction region are given by the coefficients

\[ C_{mn} = \frac{1}{2} \sin \left( \frac{2m + 1}{2} \right) \]

except for the strength of the true image which remains

\[ C_{1,0} = \frac{1}{2} \sin \left( \frac{1}{2} \right) \]

We see that the false image die-off again varies as a sinc function, but is accelerated by a factor of 2 in the sinc function argument. This behavior corresponds with our intuition. The finer we sample the hologram, the more exact our reconstruction and the more nearly it approaches the best attainable value, i.e. that for a continuous CGH.

**Fourier Domain Phase Quantization**

Fourier domain phase quantization, restriction of the phase to a discrete set of values, is of interest for several reasons. In computer holography phase quantization comes about from the infinite increment in Calcomp plotter holograms, the finite number of grey levels in kinoforms, and the finite number of masks in deposited dielectric holograms.
Various theories relating to phase quantization have been developed. We look here at the theory of Goodman and Silvestri. In this theory the effects of phase quantization are exhibited as a superposition of extraneous images in the reconstruction. A few experimental illustrations of this superposition have been generated by means of Calcomp plotter holograms.

We present here a derivation of the Goodman and Silvestri theory that is relatively compact and exhibits a mathematical technique useful in other parts of signal processing, e.g., other quantization effects and nonlinear problems. In this derivation, notation was chosen to conform to that of Goodman and Silvestri.

The object has complex amplitude $u(x)$ and Fourier spectrum $U(\xi) = A(\xi)\exp[i\phi(\xi)]$.

Quantizing $\phi(\xi)$ into N bins gives $\hat{\phi}(\xi)$, $\hat{U}(\xi) = A(\xi)\exp[i\hat{\phi}(\xi)]$. The mapping of $\phi(\xi)$ into $\hat{\phi}(\xi)$ is a staircase. We periodically continue the mapping to facilitate computation.

Now we see that any function of $\Phi(\xi)$, which is in turn a function of $\phi(\xi)$ will be periodic in $\phi(\xi)$ and piecewise constant over the same intervals of $\phi(\xi)$ in which $\Phi(\xi)$ is constant. So from the piecewise constant part of the previous statement:

$$\exp[i\hat{\phi}] = \sum_{k=-\infty}^{\infty} \exp\left[\frac{2\pi ik}{N}\right] \text{rect}\left[\frac{\phi}{2\pi/N} - k\right].$$

From the periodicity we can expand $\exp[i\hat{\Phi}(\xi)]$ in a Fourier series of functions $\exp[2\pi i (m/2\pi)\Phi]$.

First we expand the rect function obtaining

$$\text{rect}\left[\frac{\Phi}{2\pi/N} - k\right] = \sum_{J=-\infty}^{\infty} (1/N)\exp[-2\pi i (J/N)k] \sin(\pi/J)\exp[i\pi\Phi].$$

Nothing that the integration limits for calculating the series coefficients limits $k$ to range 0 to $N-1$.

$$\exp[i\hat{\phi}] = \sum_{(J)} \sin(\pi/J)\exp[i\pi\Phi] \cdot \left[(1/N) \sum_{k=0}^{n-1} \exp[2\pi i (k/N)(1-J)]\right]$$

From here arriving at the final result is straightforward, i.e., note $[\ ] = 0$, except for $J-1 = 0, \pm N, \pm 2N$, etc., let $m=(J-1)/N$ so that

$$\exp[i\hat{\phi}] = \sum_{(m)} \sin(\pi/m)\exp[i(Nm+1)\phi];$$

Finally

$$\hat{U}(\xi) = A \exp[i\hat{\phi}] = \sum_{(m)} \sin(\pi/m) A \exp[i(Nm+1)\phi].$$
**Shift Effects of Fourier Domain Phase Quantization**

Shift effects occur, for instance, when the signal is encoded on a carrier,

\[ \phi \rightarrow \phi + \beta j \]

\[ \hat{U} = A \sum_{m} \sin c \left( m + \frac{1}{N} \right) e^{i(Nm+1)(\Phi + \beta j)} \]

The complex amplitude transmittance of a prism is

\[ e^{i(Nm+1)\beta j} \]

So the phase factors can be interpreted optically as prisms.

**Depth Effects of Fourier Domain Phase Quantization**

\[ \phi \rightarrow \phi - \beta \left( j^2 + k^2 \right) \]

\[ U = A \sum_{m} \sin c \left( m + \frac{1}{N} \right) e^{i(Nm+1)(\Phi - \beta(j^2 + k^2))} \]

\[ e^{i(Nm+1)\beta(j^2 + k^2)} = \exp \left[ \frac{-i\beta}{1/Nm+1} \left( j^2 + k^2 \right) \right] \]

We can define an effective focal length of this lens as \( f \propto \frac{1}{Nm+1} \). Let’s look at the ts level problem, \( N=2 \)

| \( m = 0 \) | \( f \propto \) | \( \frac{1}{1} = 1 \) |
| \( m = 1 \) | \( f \propto \) | \( \frac{1}{N+1} = \frac{1}{3} \) |
| \( m = -1 \) | \( f \propto \) | \( \frac{1}{-N+1} = \frac{1}{-1} \) |

The generalization magnitude coupled phase quantization, is potentially useful only for very small \( N \) by fine magnitude adjustment. Generally has little effect so it is not used.
Limiting the discussion to cell-oriented holograms, we have the desired image $u_{mn}$ and its discrete Fourier transform $U_{jk}$ related by

$$u_{mn} = \sum_{j=-M/2}^{(M/2)-1} \sum_{k=-N/2}^{(N/2)-1} U_{jk} \exp\left\{+2\pi i \left[\left(jm/M\right) + \left(kn/N\right)\right]\right\}$$

$$U_{jk} = A_{jk} \exp\left(i\phi_{jk}\right)$$

**Phase Matching**

**Linear Detour-Phase Error**

**Effects on Irradiance**

We consider a nonlinearity acting on the Fourier spectrum $U_{jk}$ to produce the actual post-nonlinearity reconstructed image

$$\hat{u}_{mn} = \sum_{j} \sum_{k} C_{jk}^{mn} \exp\left\{2\pi \left[\left(jm/M\right) + \left(kn/N\right)\right]\right\},$$

where we have used the circumflex to emphasize the fact that a nonlinear operation has been performed. We have included the dependence of the nonlinearity on the observed image point. Note that the post-nonlinearity coefficients are functions of the pre-nonlinearity coefficients, i.e.,

$$C_{jk}^{mn} = C_{jk}^{mn} (A_{jk} \Phi_{jk} m, n).$$

Expressing this functional relation as a mixed Fourier-Taylor series gives

$$C_{jk}^{mn} = \sum_{p=0}^{\infty} \sum_{\alpha=-\infty}^{\infty} G_{p\alpha}^{mn} (A_{jk})^p \exp\left(\imath \alpha \Phi_{jk}\right),$$

$$C_{jk}^{mn} = \frac{1}{2\pi p!} \left[ \frac{\partial^p C_{jk}^{mn}}{\partial A^p} \right]_{A=0} \exp\left(-\imath \alpha \Phi_{jk}\right) d\Phi_{jk}$$

e.g.

$$G_{11}^{mn} = \frac{1}{2\pi} \left[ \frac{\partial C_{jk}^{mn}}{\partial A} \right] e^{-i\Phi_{jk}} d\Phi_{jk}$$
The interpretation of this somewhat formidable looking series is simplified by considering the post-
nonlinearity image as composed of several component images [6.48,57]. The \((p, \alpha)\) the component image is

\[ v_{mn}^{pa} = \sum_j \sum_k V_{jk}^{pa} \exp \left\{ 2\pi i \left[ \left( jm / M \right) + \left( kn / N \right) \right] \right\} \]

where

\[ V_{mn}^{pa} = (A_{jk})^p \exp \left( i\alpha \Phi_{jk} \right) \]

Note:

\[ v_{jk}^{11} = A_{jk} e^{i\Phi_{jk}} \]

The component images are combined with the weighing functions \( G_{mn}^{pa} \) to yield the final reconstruction image

\[ \hat{u}_{mn} = \sum_p \sum_{alpha} G_{mn}^{pa} v_{mn}^{pa} \]

The component image for \( p = 1, \alpha = 1 \) is the object, or the primary image

\[ v_{mn}^{11} = u_{mn}^{11}, v_{mn}^{11} = A_{jk} e^{i\Phi_{jk}} \]

The remaining component images are false images contributing to noise in the reconstruction. The decomposition [6.141] of the post-nonlinearity image is a decomposition in complex amplitude. The intensity is observed, so that the component images may become mixed in a complicated fashion. The post-nonlinearity intensity is

\[ i_{mn} = |\hat{u}_{mn}|^2 - \sum_p \sum_{alpha} \sum_{beta} G_{mn}^{pa} \left( G_{qbeta}^{mn} \right)^* v_{mn}^{pa} v_{mn}^{qbeta} \]

To reduce this equation to a tractable form, we follow the approach of [6.49-52]. It is convenient to work with the DFT of the intensity \( i_{jk} \). This matrix is the circular autocorrelation of the post-
nonlinearity Fourier spectra

\[ I_{jk}^{mn} = \sum_p \sum_{alpha} \sum_{beta} G_{mn}^{pa} \left( G_{qbeta}^{mn} \right)^* K_{jk} (p, \alpha; q, \beta). \]

Note that the space invariance is contained in “G”. The space variance and this particular description means that \( I_{mn} \) is the DFT over indices \((j,k)\) of \( I_{jk}^{mn} \), but the inverse DFT relation does not hold. The \( K_{jk} \)
are circular cross correlations between the \((p, \alpha)\)th and \((q,\beta)\)the component image Fourier spectra

\[ K_{jk} (p, \alpha; q, \beta) = \sum_{p=M/2}^{(M/2)-1} \sum_{sigma=-N/2}^{(N/2)-1} V_{pqsigma}^{pa} \left( V_{qbeta}^{(p-j), (sigma-k)} \right)^* \]
For a DFT this cross correlation may also be expressed, exactly as

\[ K_{jk}(p, \alpha; q, \beta) = \langle A_1^p \exp(i\alpha \Phi_1) A_2^q \exp(-i\beta \Phi_2) \rangle_{jk}, \]

where \( \langle \rangle \) signifies an ensemble average over all possible circular shifts of the discrete Fourier spectrum. The subscripts refer to two points separated by a vector displacement \( j \delta \mu, k \delta \nu \). For the primary image \( p = 1 = \alpha = \beta = 1, \)

\[ K_{jk}(1, 1; 1, 1) = \sum_{m} \sum_{n} |\mu_{mn}|^2 \exp\left\{-2\pi i \left[ (jm/M) + (kn/N) \right]\right\} \]

\[ = \langle A_1 \exp(i\Phi_1) A_2 \exp(i\Phi_2) \rangle_{jk}. \]

If the object has a random diffuser applied and was reasonably large, then the real and imaginary parts of the Fourier spectrum can be considered Gaussian processes \[6.53\]. The DFT is very amenable to such a description \[6.54\]. The \( K_{jk}(p, \alpha; q, \beta) \) can then be expressed in terms of \( K_{jk}(1, 1; 1, 1) \). To compress the following expressions in this section, we define

\[ K_{jk}(1, 1; 1, 1) = B_{jk} \]

giving

\[ K_{jk}(p, \alpha; q, \beta) = \left( B_{jk} / B_{00} \right)^\alpha B_{00}^{(p+q)/2} \left[ \frac{\Gamma\left( \frac{p+\alpha}{2} + 1 \right) \Gamma\left( \frac{q+\alpha}{2} + 1 \right)}{|\alpha|!} \right]\]

\[ \times _2 F_1 \left\{ \left[ (\alpha - p)/2 \right], \left[ (\alpha - q)/2 \right]; (|\alpha|+1); \left| B_{jk} / B_{00} \right|^2 \right\} \delta_{\alpha \beta} \]

Where \(_2 F_1\) is the Gaussian hypergeometric function and is simply a power series in \( \left| B_{jk} / B_{00} \right|^2 \)

\[ _2 F_1 \left[ a, b; c; \left| B_{jk} / B_{00} \right|^2 \right] = \sum_{\gamma=0}^{\infty} g_{\gamma} \left( \left| B_{jk} / B_{00} \right|^2 \right)^\gamma. \]

\[ g_{\gamma} = \frac{\Gamma(c)/\Gamma(a)\Gamma(b)}{\Gamma(a+\gamma)\Gamma(b+\gamma)\Gamma(c+\gamma)\gamma!}. \]

Note that on inverse discrete Fourier-transforming, the \( \left( B_{jk} \right)^\alpha \) will become \( (\alpha - 1)\)-fold convolutions of the primary image intensity while the \( \left( \left| B_{jk} \right|^2 \right)^\alpha \) will become \( (\gamma - 1)\)-fold convolutions of the primary image intensity autocorrelation. We interpret a zerofold convolution as the function itself and a minus onefold convolution as a delta function. We note in passing that an object which has a fine intensity structure, i.e., whose intensity autocorrelation is approximately a delta function, will suffer very little degradation on reconstruction from the hypergeometric function. The summary result is that the post-nonlinearity image Fourier spectral autocorrelation is
$I_{jk}^{mn} = \sum_p \sum_q \sum_{\alpha} G_{\alpha p a}^{mn} \left( B_{jk} / B_{00} \right)^2 F_1 \left( \left\lceil \left( |\alpha| - p \right) / 2 \right\rceil, \left\lceil \left( |\alpha| - q \right) / 2 \right\rceil; \left( |\alpha| + 1 \right) \left| B_{jk} / B_{00} \right| / 2 \right) ;$

$B_{00}^{(p=q) / 2} \left\lceil \Gamma \left( \frac{p + |\alpha|}{2} + 1 \right) \Gamma \left( \frac{q + |\alpha|}{2} + 1 \right) \right\rceil / |\alpha| ! \right\} .

One can experimentally observe separate component images in this sum. One first picks a compound object, one consisting of some recognizable object together with a strong point. Forming the CGH with some known nonlinear degradation then gives rise to false images whose strength can be calculated from (6.151). The various convolutions and correlations then reproduce the recognizable figure through interaction of the point and figure. These reproductions will however have varying orientation and displacements from the viewing field center depending on where they originate in the sum of (6.151) [6.55]. Figure 6.14 [6.56] shows an application of this technique to studying Fourier domain phase quantization.

**Phase Nonlinearities**

For most objects, distorting the Fourier spectrum phase has stronger effects than distorting the modulus. This has been shown for objects with random diffusers [6.58] and demonstrated for other objects [6.59]. Because of the Fourier spectrum phase’s importance, we concentrate first on phase non-linearities. For phase non-linearities (6.129) reduces to

$C_{jk}^{mn} = A_{jk} \sum_{\alpha=-\infty}^{\infty} G_{\alpha p a}^{mn} \exp \left( i\alpha \Phi_{jk} \right)$

Shift-invariant phase nonlinearities have the simpler form

$C_{jk} = A_{jk} \sum_{\alpha=-\infty}^{\infty} G_{\alpha} \exp \left( i\alpha \Phi_{jk} \right)$

$G_{\alpha} = \frac{1}{2\pi} \int_{-\pi}^{\pi} C_{jk} \exp \left( -i\alpha \phi_{jk} \right) d\phi_{jk}$

**Fourier Domain Phase Quantization**

$u_{mn} = \sum_j \sum_k U_{jk} e^{2\pi i \left( \frac{jm}{M} \frac{kn}{N} \right)}$

$u_{mn} = \sum_j \sum_k C_{jk}^{mn} e^{i \left( \frac{jm}{M} \frac{kn}{N} \right)}$

$C_{jk}^{mn} = C_{jk} \left( A_{jk}, \Phi_{jk}, m, n \right) = \sum_{p=0}^{\infty} \sum_{\alpha=-\infty}^{\infty} G_{\alpha p a}^{mn} \left( A_{jk} \right)^p e^{i\alpha \Phi_{jk}}$
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For point nonlinearities in the Fourier Domain

\[ C_{jk}^{mn} = C_{jk} = \hat{A}_{jk} e^{i\Phi_{jk}} \]

\[ C_{jk}^{mn} = C_{jk} = \sum_{p=0}^{\infty} \sum_{\alpha=-\infty}^{\infty} G_{p\alpha} A_{jk}^p e^{i\alpha \Phi_{jk}} \]

For point phase nonlinearities

\[ C_{jk} = A_{jk} \sum_{\alpha=-\infty}^{\infty} G_{\alpha} e^{i\Phi_{jk}} \]

or

\[ e^{i\Phi} \sum_{\alpha=-\infty}^{\infty} G_{\alpha} e^{i\alpha \Phi_{jk}} \]

\[ \hat{I}_{jk} = \sum_{p} \sum_{q} \sum_{\alpha} G_{mn}^{pq} \left( G_{q\alpha}^{mn} \right)^* \cdot \left[ \frac{\Gamma \left( \frac{p+|\alpha|}{2} + 1 \right) \Gamma \left( \frac{q+|\alpha|}{2} + 1 \right)}{\alpha!} \right] \cdot \frac{\Gamma \left( |\alpha| + 1 \right)}{\Gamma \left( \frac{|\alpha| - p}{2} \right) \Gamma \left( \frac{|\alpha| - q}{2} \right)} \cdot \frac{\Gamma \left( \frac{|\alpha| - p + \gamma}{2} \right) \Gamma \left( \frac{|\alpha| - q + \gamma}{2} \right)}{\Gamma \left( |\alpha| + 1 + \gamma \right) \gamma!} \cdot \left( \frac{1}{B_{00}^2} \right) \left( \frac{1}{|B_{00}| 2\gamma} \right) B_{00}^{(p+q)/2} |B_{jk}|^{2\gamma} B_{jk}^\alpha \]