COMPUTER GENERATED HOLOGRAMS
Optical Sciences 627 -- William J. Dallas

PART I: CHAPTER ONE
THE FOURIER TRANSFORM

Additional Reading

- J.D. Gaskill, **Linear Systems, Fourier Transforms, and Optics**, Wiley, New York, 1978.
- R.N. Bracewell, **The Fourier Transform and its Applications** (3rd edition, revised), McGraw-Hill, New York, 1999.
- J.W. Goodman, **Introduction to Fourier Optics** (2nd Edition), McGraw-Hill, New York, 1996.

The Greek Alphabet

Alpha	α, A		Nu	ν, N
Beta	β, B		Xi	ξ, Ξ
Gamma	γ, Γ		Omicron	o, O
Delta	δ, Δ		Pi	π, Π
Epsilon	ε, E		Rho	ρ, P
Zeta	ζ, Z		Sigma	σ, Σ
Eta	η, N		Tau	τ, T
Theta	θ, Θ		Upsilon	υ, Y
Iota	ι, I		Phi	ϕ, Φ
Kappa	κ, K		Chi	χ, X
Lambda	λ, Λ		Psi	ψ, Ψ
Mu	μ, M		Omega	ω, Ω

Introduction

The most common abbreviation we will use is CGH for Computer Generated Hologram.

The Fourier transform forms the mathematical foundation of computer generated holography. We will define the Fourier transform in one and two dimensions. Most of the CGH mathematics is in two dimensions; one-dimensional examples are used for simplicity. In the body of this chapter, we will also explore Fourier series, the discrete Fourier transform or DFT, and generalized harmonic analysis. We begin this chapter with the definitions of several varieties of Fourier transform including: the continuous Fourier transform, Fourier series, and the discrete Fourier transform or DFT.

1-D Continuous Fourier Transform

We will abbreviate Fourier transform by F.T. and by $\mathscr{F}$. The inverse Fourier transform will be abbreviated by I.F.T and by $\mathscr{F}^{-1}$. Note that the inverse Fourier transform is an expression of the original function as a "weighted sum" of exponentials with simple arguments. The phases of the exponentials are linear in the position coordinate. The exponential is therefore called a linear phase factor. Another name comes from the Euler relation $\exp(i\phi) = \cos(\phi) + i\sin(\phi)$: the function $\exp(i\phi)$ is called a cisoid in analogy to a sinusoid. More exactly: . An example of such a factor is $\exp(-2\pi i\xi_0 x)$, where ξ_0 is a constant. This decomposition is useful in optics because the exponential functions behave in a simple, well understood way as does the re-composition of the functions after they have been acted upon by an optical system.

Forward Fourier Transform (**F.T.**): $$U(\xi) = \int_{-\infty}^{\infty} u(x) e^{-2\pi i\xi x} dx$$

Inverse Fourier Transform: (**I.F.T**) $$u(x) = \int_{-\infty}^{\infty} U(\xi) e^{+2\pi i\xi x} d\xi$$

The Fourier transform is a method for finding the weights of the linear-phase exponentials. Notice that we are using frequency instead of angular frequency in the argument of the exponential. This convention eliminates normalization factors in the Fourier relations.

2-D Continuous Fourier Transform

The extension to two dimensions is straight forward. Notice that the exponential functions are separable. Using the following definitions, we can express the arguments as a scalar product of 2-dimensional vectors.

F.T. $$U(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y) e^{-2\pi i(\xi x+\eta y)} dxdy$$

I.F.T. $$u(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\xi,\eta) e^{+2\pi i(\xi x+\eta y)} d\xi d\eta$$

Expressing the coordinates in vector form makes the expression of these relations somewhat more compact, and allows for easy extension to higher dimensions. Vectors are bolded. Unit vectors have a karat.

Let $\mathbf{r} = x\ \hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$; $\boldsymbol{\rho} = \xi\ \hat{\mathbf{x}} + \eta\,\hat{\mathbf{y}}$; $\boldsymbol{\rho}\bullet\mathbf{r} = \xi\ x + \eta\ y$, so that

F.T. $$U(\boldsymbol{\rho}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\mathbf{r}) e^{-2\pi i\boldsymbol{\rho}\bullet\mathbf{r}} d^2 r$$

I.F.T. $$u(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\boldsymbol{\rho}) e^{+2\pi i(\boldsymbol{\rho}\bullet\mathbf{r})} d^2\rho$$

Fourier Series and Two Related Definitions

Fourier Series in 2-D

The Fourier series expresses a periodic function as a sum of linear phase factors. The forward transformation for calculating the Fourier series coefficients and the inverse transformation, the Fourier series which reconstitutes the original function, is

Fourier Series Coefficients: $U_{mn} = \Delta\xi\Delta\eta \int_{-\frac{1}{2\Delta\xi}}^{\frac{1}{2\Delta\xi}} \int_{-\frac{1}{2\Delta\eta}}^{\frac{1}{2\Delta\eta}} u(x,y) e^{-2\pi i(m\Delta\xi x + n\Delta\eta y)} dx\, dy$

Fourier Series: $u(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=\infty}^{\infty} U_{mn} e^{2\pi i(m\Delta\xi x + n\Delta\eta y)}$

The object is periodic: $u(x,y) = u\left(x - \frac{m}{\Delta\xi}\ ,\ y - \frac{n}{\Delta\eta}\right)$.

There is a correspondence between the Fourier transform and the Fourier series that reflects the periodicity of the function that is being transformed. The Fourier transform of a periodic function consists of regularly spaced points that are represented by Dirac delta functions. The weights of these points are proportional to the Fourier coefficients.

Generalized Harmonic Analysis

Generalized harmonic analysis is a fancy name for a simple procedure. In essence, we expand a function as a Fourier series. We then substitute a function for a variable in the argument. We restrict our attention to one dimension for simplicity and begin with the usual 1-D Fourier series. Consider the Fourier series

$$u(x) = \sum_{m=-\infty}^{\infty} U_m e^{2\pi i m \xi_0 x}$$

At this point the key substitution is made and it is important to realize that the Fourier series relation is a point relation, i.e., the "x" in the exponential corresponds point-wise to the "x" in the argument of the function.

Therefore, we can substitute for that "x", a function "f of x", giving the expansion

$$u[f(x)] = \sum_{m=-\infty}^{\infty} U_m e^{2\pi i m \xi_0 f(x)}$$

More important use of generalized harmonic analysis is to discuss small distortions of regular gratings. Making the substitution

$$f(x) = x + \frac{\phi(x,y)}{2\pi\xi_0}$$

we see that the regular grating has been slightly distorted by the addition of a spatially varying phase. We will discuss the effects of this distortion in more detail later, but for now we observe that

$$u[f(x)] = u\left[x + \frac{\phi(x,y)}{2\pi}\right] = \sum_{m=-\infty}^{\infty} U_m e^{2\pi i m \xi_0 x} e^{im\phi(x,y)}$$

The spatial distortion has been converted into a phase.

Circular Harmonic Decomposition

Another use for the Fourier series is circular harmonic decomposition of a 2-D function. It is basically a Fourier-series decomposition in angle. If we start with the function $f_c(x,y)$, where the subscript *c* stands for Cartesian, and the usual relations between the Cartesian and polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta \quad r = \sqrt{x^2+y^2} \quad \theta = \arctan\left(\frac{y}{x}\right)$$

$$f_c(x,y) = f(r,\theta)$$

The circular harmonic decomposition is

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)\exp(in\theta)$$

The radial coefficients are given by the usual Fourier-series-coefficient formula

$$f_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\theta)e^{-in\theta}d\theta$$

Note that the Fourier coefficients in this case are not constants, but rather functions of *r*.

Discrete Fourier Transform (DFT)

The correspondence of punctility and periodicity can be extended to periodic objects which consist of regular arrays of points. From the previous section, we see that the continuous Fourier transform of such objects will itself be periodic and consist of points. The relation between the point strengths of one period in the object and one period in the Fourier transform is given by the discrete Fourier transform:

DFT

Forward DFT: $$U_{jk} = \frac{1}{\sqrt{M}}\frac{1}{\sqrt{N}}\sum_{m=-\frac{M}{2}}^{\frac{M}{2}-1}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} u_{mn}\, e^{-2\pi i\left(\frac{jm}{M}+\frac{kn}{N}\right)}$$

Inverse DFT: $$u_{mn} = \frac{1}{\sqrt{M}}\frac{1}{\sqrt{N}}\sum_{j=-\frac{M}{2}}^{\frac{M}{2}-1}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} U_{jk}\, e^{2\pi i\left(\frac{jm}{M}+\frac{kn}{N}\right)}$$

We note that the DFT is a discrete-to-discrete transformation. There are several variations on defining the DFT. One variation is in the normalization factor another in the range covered by the indices. This index variability is of practical importance.

FFT

When a Fourier transform is done on a computer, many times people talk about performing a fast Fourier transform or FFT. The FFT is actually an algorithm for performing the discrete Fourier transform or DFT. The algorithm speeds computation of the DFT by a factor $\frac{n}{\log_2 n}$ which can be considerable for large transforms.

We will use the term FFT to spotlight a difference in the location of the origin for the transform from our definition of the discrete Fourier transform.

Quadrant-exchange and Checkerboarding

Most libraries of computer algorithms define the indices going from zero to an upper limit whereas we define an almost symmetric index range about zero. These differences require a technique variously known as quadrant swapping or checker-boarding in order to bring consistency into calculations.

In one dimension our definition of the DFT is: $U_j = \frac{1}{\sqrt{M}} \sum_{m=-\frac{M}{2}}^{\frac{M}{2}-1} u_m e^{-2\pi i \frac{j m}{M}}$

the "FFT" definition is : $V_j = \frac{1}{M} \sum_{n=0}^{M-1} \mathrm{v}_n e^{-2\pi i \frac{n j}{M}}$.

In Appendix E we show that $\sqrt{M}\left(-1\right)^j V_j = U_j$. Upon performing the discrete Fourier transform, a striping or multiplication by alternating plus and minus ones, in the transform, occurs when the indices are shifted from our definition to the computer algorithm definition. In 2-dimensions, the striping becomes checker boarding. The checker-boarding, i.e. multiplication by the ±1 checkerboard in the transform, is equivalent to exchanging quadrants in the object. The quadrants that are exchanged are the first with the third and the second with the fourth. The actual correspondence between the two operations is

- Post-transform checker-boarding = pre-transform quadrant-exchange
- Post-transform quadrant-exchange = pre-transform checker-boarding

The usual procedure is: DFT = quadrant-exchange → FFT → quadrant-exchange.

APPENDIX A
THREE- AND FOUR- DIMENSIONAL FOURIER TRANSFORM

3-D Fourier Transform

Extension to higher dimensions is natural in the vector form. The equation is the same but the vectors have higher dimensions. We write out the 3-dimensional expression explicitly because we will use it later in discussing wave propagation.

F.T. $$U\left(\xi,\eta,\zeta\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\left(x,y,z\right) e^{-2\pi i\left(\xi x+\eta y+\zeta z\right)} dxdydz$$

I.F.T. $$u\left(x,y,z\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U\left(\xi,\eta,\zeta\right) e^{+2\pi i\left(\xi x+\eta y+\zeta z\right)} d\xi d\eta d\zeta$$

Putting the coordinates in vector form we have

$$\mathbf{r} = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}} \quad ; \quad \boldsymbol{\rho} = \xi\ \hat{\mathbf{x}} + \eta\ \hat{\mathbf{y}} + \zeta\ \hat{\mathbf{z}} \quad ; \quad \boldsymbol{\rho} \bullet \mathbf{r} = \xi\ x + \eta\ y + \zeta\ z$$

and the Fourier transform is

F.T. $$U\left(\vec{\rho}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\left(\vec{r}\right) e^{-2\pi i \vec{\rho}\bullet\vec{r}} d^3 r$$

With the corresponding inverse transform being

I.F.T. $$u(\vec{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\vec{\rho}) e^{+2\pi i(\vec{\rho}\bullet\vec{r})} d^3\rho$$

APPENDIX B
THREE- AND FOUR- DIMENSIONAL VECTOR FOURIER TRANSFORM

3-D Vector Fourier Transform

So far we have been dealing with functions that are scalars. They may also be vectors.

$$\mathbf{u}(\mathbf{r}) = u_x(x,y,z)\,\hat{\mathbf{x}} + u_y(x,y,z)\,\hat{\mathbf{y}} + u_z(x,y,z)\,\hat{\mathbf{z}}$$

The Fourier transform is then

F.T. $$\mathrm{U}(\boldsymbol{\rho}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{u}(\mathbf{r}) e^{-2\pi i(\boldsymbol{\rho}\bullet\mathbf{r})} d^3 r$$

I.F.T. $$\mathbf{u}(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{U}(\boldsymbol{\rho}) e^{+2\pi i(\boldsymbol{\rho}\bullet\mathbf{r})} d^3\rho$$

Each component is transformed independently of the others. The Fourier transformation of vector fields will prove useful in dealing with electric and magnetic fields.

APPENDIX C
N-DIMENSIONAL VECTOR FOURIER TRANSFORM

In extending the Fourier transform to higher dimensions, it is useful to replace the explicit sums in the exponentials by scalar products in vector notation. For comparison, in two dimensions, the correspondences are:

$$r_1 = x \;\; ; r_2 = x \;\; ; \rho_1 = \xi \;\; ; \rho_2 = \eta$$
$$\rho \cdot r = \rho_1 r_1 + \rho_2 r_2 = \xi x + \eta y$$

The extension of the Fourier transform and its inverse to "n" dimensions is:

F.T. $$\mathrm{U}(\boldsymbol{\rho}) = \int_n \mathbf{u}(\mathbf{r}) e^{-2\pi i(\boldsymbol{\rho}\bullet\mathbf{r})} d^n r$$

I.F.T. $\mathbf{u}(\mathbf{r}) = \int_n \mathbf{U}(\boldsymbol{\rho}) e^{+2\pi i(\boldsymbol{\rho}\cdot\mathbf{r})} d^n \rho$

APPENDIX D
THE FOURIER TRANSFORM IN POLAR COORDINATES

For some CGH types, polar coordinates are more convenient than Cartesian coordinates. The equations for the conversion from polar to Cartesian and the inverses in object and Fourier spaces are

$$x = r\cos\theta \quad y = r\sin\theta \quad r = \sqrt{x^2 + y^2} \quad \theta = \arctan\left(\frac{y}{x}\right)$$

$$\xi = \rho\cos\psi \quad \eta = \rho\sin\psi \quad \rho = \sqrt{\xi^2 + \eta^2} \quad \psi = \arctan\left(\frac{\eta}{\xi}\right)$$

We begin with the Fourier transform in two dimensions and Cartesian coordinates.

F.T. $U(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(\xi x + \eta y)} u(x,y)\, dx dy$

I.F.T. $u(x,y) = \int\int e^{2\pi i(\xi x + \eta y)} U(\xi,\eta)\, d\xi d\eta$

Next we write the equation in a mixed Cartesian-polar form where we use the circular harmonic decomposition $u_c(x,y) = u(r,\theta) = \sum_{n=-\infty}^{\infty} u_n(r) e^{in\theta}$. The temporary form is

$$U(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(\xi x + \eta y)} \sum_{n=-\infty}^{\infty} u_n(r) e^{in\theta} dx dy$$

In polar form

$$U(\rho,\psi) = \int_{r=0}^{\infty}\int_{\theta=-\pi}^{\pi} e^{-2\pi i r\rho\cos(\theta-\psi)} \sum_{n=-\infty}^{\infty} u_n(r) e^{in\theta} r dr d\theta$$
$$= \sum_{n=-\infty}^{\infty}\int_{r=0}^{\infty} u_n(r) e^{-in\psi} r \left[\int_{\theta=-\pi}^{\pi} e^{-2\pi i r\rho\cos\theta} e^{in\theta} d\theta\right] dr$$

At this point, is handy to introduce the Bessel function. The most useful definition of the Bessel function of the first kind (*J*) and n^{th} order is

$$J_n(2\pi r\rho) = \frac{i^n}{2\pi}\int_{-\pi}^{\pi} e^{-2\pi i r\rho\cos\theta} e^{in\theta} d\theta$$

We have that

$$U(\rho,\psi)=\int_{r=0}^{\infty}\int_{\theta=-\pi}^{\pi} e^{-2\pi i r\rho\cos(\theta-\psi)}\sum_{n=-\infty}^{\infty}u_n(r)e^{in\theta}drd\theta$$
$$=2\pi\sum_{n=-\infty}^{\infty}(-i)^n e^{in\psi}\int_{r=0}^{\infty}u_n(r)rJ_n(r\rho)dr$$

Hankel transform

The n^{th} order Hankel transform of a function is

$$H_n(\rho)=2\pi\int_{r=0}^{\infty}u_n(r)rJ_n(2\pi r\rho)dr$$

So that, if we define the following, closely related function

$$U_n(\rho)=(-i)^n H_n(\rho)=2\pi(-i)^n\int_{r=0}^{\infty}u_n(r)rJ_n(2\pi r\rho)dr$$

we have

$$U(\rho,\psi)=\sum_{n=-\infty}^{\infty}U_n(\rho)e^{in\psi}$$

This is the circular harmonic decomposition of the 2-D Fourier transform. The radial coefficient functions are related to the corresponding functions in direct space by the Hankel transform. Equally as interesting, is the fact that the circular harmonic orders retain their identities under two-dimensional Fourier transformation. There is no "cross-talk" among the circular harmonic components of a 2-D function when it is 2-D Fourier transformed. In summary

$$u(r,\theta)=\sum_{n=-\infty}^{\infty}u_n(r)e^{in\theta}\qquad U_n(\rho)=2\pi(-i)^n\int_{r=0}^{\infty}u_n(r)rJ_n(2\pi r\rho)dr\qquad U(\rho,\psi)=\sum_{n=-\infty}^{\infty}U_n(\rho)e^{in\psi}$$

For Radially Symmetric Objects

The Bessel function definition gives the interesting Fourier series expansion

$$e^{-2\pi i r\rho\cos\theta(\Psi-\theta)}=2\pi\sum_{m=-\infty}^{\infty}(-i)^{m}J_m(2\pi r\rho)e^{im}\theta$$

This expansion is useful when analyzing phase holograms and when dealing with Fourier transforms of radially-symmetric functions, because the symmetry destroys all but the $m=0$ term in the expansion. The relation that follows is for radially symmetric 2-dimensional functions, i.e., functions that are completely described by their behavior along a radius. In earlier optical literature, this relation is referred to as the Fourier-Bessel transform. The more modern practice is to refer to it as the zero-order Hankel transform. The validity of this formula is based on the definition of the Bessel function and the fact that the angular integration in the Fourier transform does not involve the object for radially symmetric objects.

$$U(\rho)=2\pi\int_0^{\infty}u(r)J_0(2\pi\rho r)rdr$$

$$u(r) = 2\pi \int_0^\infty U(\rho) J_0\left(2\pi\rho r\right) \rho d\rho$$

APPENDIX E
FOURIER TRANSFORM IN SPHERICAL COORDINATES

In 3-dimensions, the Fourier transform is

$$\boldsymbol{\rho} = \xi\hat{\mathbf{x}} + \eta\hat{\mathbf{y}} + \zeta\hat{\mathbf{z}} \quad \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$U(\boldsymbol{\rho}) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty} u(\mathbf{r})e^{-2\pi i\mathbf{r}\cdot\boldsymbol{\rho}}d^3r$$

The Cartesian to spherical coordinate transformations are

$$x = r\cos\theta\sin\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\phi$$

$$\xi = \rho\cos\psi\sin\chi \quad \eta = \rho\sin\psi\sin\chi \quad \zeta = \rho\cos\chi$$

The Fourier transform conversion begins with

$$U(\rho,\psi,\chi) = U_c(\xi,\eta,\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_c(x,y,z)e^{-2\pi i(\xi x+\eta y+\zeta z)}dxdydz =$$

$$\int_{r=0}^{\infty}\int_{\theta=-\pi}^{\pi}\int_{\phi=0}^{\pi} u(r,\theta,\varphi)e^{-2\pi i r\rho(\cos\psi\sin\chi\cos\theta\sin\varphi+\sin\psi\sin\chi\sin\phi\sin\varphi+\cos\varphi\cos\chi)}r^2\sin\varphi drd\theta d\varphi$$

With some simple trigonometric manipulations we can simplify the exponent.

$$\cos\psi\sin\chi\cos\theta\sin\varphi + \sin\psi\sin\chi\sin\theta\sin\varphi + \cos\chi\cos\varphi =$$
$$\cos(\psi-\theta)\sin\chi\sin\varphi + \cos\chi\cos\varphi$$

At this stage we can see if the object is circularly symmetric, i.e., independent of φ, then the θ integral is similar to the one we used in polar coordinates. For circularly symmetric objects:

$$U(\rho,\chi) = 2\pi\int_{r=0}^{\infty}\int_{\varphi=0}^{\pi} u(r,\varphi)J_0(2\pi r\rho\sin\chi\sin\varphi)e^{-2\pi i r\rho\cos\chi\cos\varphi}r^2\sin\varphi drd\varphi$$ For spherically symmetric objects we have the choice of either using the relations for circularly symmetric objects and performing the $\boldsymbol{\varphi}$ integration or going back the original Fourier transform relation in spherical coordinates. In either case we note that the Fourier transform will also be spherically symmetric so that we have the freedom of choosing a particular axis. The choice that simplifies the calculations is to let $\chi = 0$. This leaves us to do the integral

$$U(\rho) = 2\pi\int_{r=0}^{\infty} u(r)r^2\left[\int_{\varphi=0}^{\pi} e^{-2\pi i\rho\cos\varphi}\sin\varphi d\varphi\right]dr$$

With the variable change

$$p = -\cos\phi \text{ and } dp = \sin\phi d\phi$$

the integral in brackets becomes

$$\int_{\varphi=0}^{\pi} e^{-2\pi i\rho\cos\varphi}\sin\varphi d\varphi = \int_{p=-1}^{1} e^{2\pi i\rho p}dp = \frac{1}{2\pi i\rho}\left(e^{2\pi i\rho} - e^{-2\pi i\rho p}\right) = \frac{\sin(2\pi\rho)}{\pi\rho} = 2\,\mathrm{sinc}(2\rho)$$

can be done by inspection to give the final result

$$U(\rho) = 4\pi\int_0^\infty u(r)r^2\,\mathrm{sinc}(2\rho r)dr$$

where $\mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$.

APPENDIX F
FOURIER TRANSFORM IN HYPERSPHERICAL COORDINATES

For higher-dimensional radially-symmetric functions, the relation is

$$F(\rho) = \frac{2\pi}{\rho^{\frac{n}{2}-1}}\int_0^\infty f(r)J_{\frac{n}{2}-1}(2\pi\rho r)r^{\frac{n}{2}}dr$$

In order to recover the 3-D relations from the general relation, it is important to know that

$$J_{\frac{1}{2}} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sin x \qquad ; \qquad J_{-\frac{1}{2}} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\cos x$$

APPENDIX G
RELATION OF THE FFT TO THE DFT
(See Also Part I Chapter Two – Comb Math)

In one dimension our definition of the DFT is $U_j = \dfrac{1}{\sqrt{M}}\displaystyle\sum_{m=-\frac{M}{2}}^{\frac{M}{2}-1} u_m e^{-2\pi i\frac{jm}{M}}$

the "FFT" definition is $V_j = \dfrac{1}{M}\displaystyle\sum_{n=0}^{M-1} \mathrm{v}_n e^{-2\pi i\frac{nj}{M}}$

let $n = m + \frac{M}{2}$ so that

$$V_j = \frac{1}{M}\sum_{m=-\frac{M}{2}}^{\frac{M}{2}-1} \mathrm{v}_{m+\frac{M}{2}}\, e^{-2\pi i\frac{\left(m+\frac{M}{2}\right)j}{M}} = \frac{e^{i\pi j}}{M}\sum_{m=-\frac{M}{2}}^{\frac{M}{2}-1} \mathrm{v}_{m+\frac{M}{2}}\, e^{-2\pi i\frac{m\,j}{M}} = \frac{(-1)^j}{M}\sum_{m=-\frac{M}{2}}^{\frac{M}{2}-1} \mathrm{v}_{m+\frac{M}{2}}\, e^{-2\pi i\frac{m\,j}{M}}$$

Collecting the left-most and right-most members of the equation chain, we have the more readable relation:

$$V_j = \frac{(-1)^j}{M}\sum_{m=-\frac{M}{2}}^{\frac{M}{2}-1} \mathrm{v}_{m+\frac{M}{2}}\, e^{-2\pi i\frac{m\,j}{M}}$$

now if we put our object into the array such that $u_m = \mathrm{v}\left(m+\frac{M}{2}\right)$ we get

$$V_j = \frac{(-1)^j}{M}\sum_{m=-\frac{M}{2}}^{\frac{M}{2}-1} u_m\, e^{-2\pi i\frac{m\,j}{M}}$$

The result is that $\sqrt{M}\,(-1)^j\, V_j = U_j$.

APPENDIX H
THE FRESNEL TRANSFORM

Introduction

The Fresnel transform is useful in discussing wave propagation. It is also quite convenient to calculate when one notes that it can be related to the Fourier transform. The definition of the Fresnel transform is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{+i\pi(\xi-x)^2} dx$$

By multiplying out the quadratic phase factor, we can reorganize the calculation as follows:

$$\hat{f}(\xi) = e^{+i\pi\xi^2}\int_{-\infty}^{\infty}\left[\, f(x)\, e^{+i\pi x^2}\right] e^{-2\pi i\xi x} dx$$

We then see that the Fresnel transform is simply a Fourier transform with the inclusion of the pre- and post-multiplication by quadratic phase factors. The corresponding inversion formula is

$$f(x) = e^{-i\pi x^2}\int_{-\infty}^{\infty}\left[\hat{f}(\xi)\, e^{-i\pi\xi^2}\right] e^{+2\pi i\xi x}\, d\xi$$

APPENDIX I

THE UNCERTAINTY RELATION

Introduction

When working with functions and their Fourier transforms, it is convenient to consider the functions as defining probability density functions. If we consider a function $f(x)$, the associated probability density function is defined by

$$\frac{|f(x)|^2}{\int_{-\infty}^{\infty}|f(x)|^2\,dx}$$

The average, expectation value, of a quantity $q(x)$ is

$$\langle q(x)\rangle=\int_{-\infty}^{\infty}q(x)\frac{|f(x)|^2}{\int_{-\infty}^{\infty}|f(x_0)|^2\,dx_0}dx$$

We now apply this interpretation to an examination of the relationship between the widths of a function and its Fourier transform. We have seen that the scale of a function and of its Fourier transform are inversely proportional to one another. The uncertainty relation gives a precise lower bound for the product of the function's width and that of its Fourier transform. This relation is a classical analog of the Heisenberg uncertainty principle from quantum mechanics. For purposes of the uncertainty relation, the spatial extent of a function is specified by its RMS width, or standard deviation which is the root of the variance

$$\left\langle(x-\overline{x})^2\right\rangle=\int_{-\infty}^{\infty}(x-\overline{x})^2\frac{|f(x)|^2}{\int_{-\infty}^{\infty}|f(x_0)|^2\,dx_0}dx$$

The mean is

$$\overline{x}=\langle x\rangle=\int_{-\infty}^{\infty}x\frac{|f(x)|^2}{\int_{-\infty}^{\infty}|f(x_0)|^2\,dx_0}dx$$

We will set the mean to zero in what follows. The uncertainty principle states that

$$\sqrt{\left\langle x^2\right\rangle}\sqrt{\left\langle \xi^2\right\rangle}\geq\frac{1}{4\pi}$$

In order to demonstrate the validity of this relation, we will need two mathematical tools: Parseval's Theorem and the Schwartz Inequality. A third relation, the derivative of the absolute value squared of a function, will prove useful. These three relations are found at the end of this appendix.

Proceeding, we write out the variance product as

$$\left\langle x^2\right\rangle\left\langle \xi^2\right\rangle=\frac{\int_{-\infty}^{\infty}x^2\left|f(x)\right|^2dx\int_{-\infty}^{\infty}\xi^2\left|F(\xi)\right|^2d\xi}{\int_{-\infty}^{\infty}\left|f(x)\right|^2dx\ \int_{-\infty}^{\infty}\left|F(\xi)\right|^2d\xi}$$

Operating on the numerator of the second factor on the right we apply the derivative transformation formula and Parseval's theorem to obtain

$$\int_{-\infty}^{\infty}\xi^2\left|F(\xi)\right|^2d\xi=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\left|\dot{f}(x)\right|^2dx$$

where

$$\dot{f}(x)=\frac{d}{dx}f(x)$$

At this point we have

$$\left\langle x^2\right\rangle\left\langle \xi^2\right\rangle=\frac{\int_{-\infty}^{\infty}x^2\left|f(x)\right|^2dx\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\left|\dot{f}(x)\right|^2dx}{\int_{-\infty}^{\infty}\left|f(x)\right|^2dx\ \int_{-\infty}^{\infty}\left|F(\xi)\right|^2d\xi}=\frac{1}{4\pi^2}\frac{\int_{-\infty}^{\infty}x^2\left|f(x)\right|^2dx\int_{-\infty}^{\infty}\left|\dot{f}(x)\right|^2dx}{\int_{-\infty}^{\infty}\left|f(x)\right|^2dx\ \int_{-\infty}^{\infty}\left|F(\xi)\right|^2d\xi}$$

With application of the Parseval relation to the denominator, we obtain

$$\left\langle x^2\right\rangle\left\langle \xi^2\right\rangle=\frac{1}{4\pi^2}\frac{\int_{-\infty}^{\infty}x^2\left|f(x)\right|^2dx\int_{-\infty}^{\infty}\left|\dot{f}(x)\right|^2dx}{\int_{-\infty}^{\infty}\left|f(x)\right|^2dx\ \int_{-\infty}^{\infty}\left|f(x)\right|^2dx}=\frac{1}{4\pi^2}\frac{\int_{-\infty}^{\infty}x^2\left|f(x)\right|^2dx\int_{-\infty}^{\infty}\left|\dot{f}(x)\right|^2dx}{\left[\int_{-\infty}^{\infty}\left|f(x)\right|^2dx\ \right]^2}$$

Now we rearrange the equation slightly to get

$$\begin{aligned}4\pi^2\left\langle x^2\right\rangle\left\langle \xi^2\right\rangle\left(\int_{-\infty}^{\infty}\left|f(x)\right|^2dx\right)^2&=\int_{-\infty}^{\infty}x^2\left|f(x)\right|^2dx\int_{-\infty}^{\infty}\left|\dot{f}(x)\right|^2dx\\&=\int_{-\infty}^{\infty}xf(x)xf^*(x)dx\int_{-\infty}^{\infty}\dot{f}(x)\dot{f}^*(x)dx\end{aligned}$$

Next we apply the Schwartz inequality with

$$p(x)=xf(x)$$

and

$$q(x)=\dot{f}(x)$$

obtaining

$$4\pi^2\left\langle x^2\right\rangle\left\langle \xi^2\right\rangle\left(\int_{-\infty}^{\infty}\left|f(x)\right|^2dx\right)^2\geq\left[\frac{1}{2}\int_{-\infty}^{\infty}xf(x)\dot{f}^*(x)+xf^*(x)\dot{f}^*(x)dx\right]^2$$

The differentiation relation at the end of this appendix allows us to rewrite the integrand on the right as

$$4\pi^2\left\langle x^2\right\rangle\left\langle \xi^2\right\rangle\left(\int_{-\infty}^{\infty}\left|f(x)\right|^2dx\right)^2\geq\left[\frac{1}{2}\int_{-\infty}^{\infty}x\frac{d}{dx}\left|f(x)\right|^2dx\right]^2$$

The final step is to integrate the right side by parts to get

$$4\pi^2 \langle x^2\rangle\langle \xi^2\rangle\left(\int_{-\infty}^{\infty}|f(x)|^2\,dx\right)^2 \geq \left[xf(x)\int_{-\infty}^{\infty}-\frac{1}{2}\int_{-\infty}^{\infty}|f(x)|^2\,dx\right]^2$$

The first term on the right side of the equation is zero for functions vanishing at infinity. Rearranging the remaining terms gives

$$\langle x^2\rangle\langle \xi^2\rangle \geq \frac{1}{16\pi^2}$$

or

$$\sqrt{\langle x^2\rangle}\sqrt{\xi^2} \geq \frac{1}{4\pi}$$

which is the desired result.

Parseval's Theorem

Parseval's relation states that:

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}|F(\xi)|^2\,d\xi$$

Proof:

$$\int_{-\infty}^{\infty}|F(\xi)|^2\,d\xi = \int_{-\infty}^{\infty}F(\xi)F^*(\xi)d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x_0)f^*(-x_1)e^{2\pi i(x_0+x_1)\xi}dx_0dx_1d\xi$$

we have that

$$\int_{-\infty}^{\infty}|F(\xi)|^2\,d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x_0)f^*(-x_1)e^{2\pi i(x_0+x_1)\xi}dx_0dx_1d\xi$$

or

$$\int_{-\infty}^{\infty}|F(\xi)|^2\,d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x_0)f^*(-x_1)\delta(x_0+x_1)dx_0dx_1$$

which immediately gives us the desired relation.

The Schwartz Inequality

The next element of the derivation is the Schwartz inequality.

$$\left[\frac{1}{2}\int_a^b p(x)q^*(x)+p^*(x)q(x)dx\right]^2 \leq \int_a^b|p(x)|^2dx\int_a^b|q(x)|^2dx$$

This inequality is familiar from trigonometry as the triangle inequality. It states that the sum of the lengths of two sides of a triangle is equal to or exceeds the length of the third side. Consider a triangle with sides A, B, and C. We represent it by the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. The triangle inequality states that

$$\left(|A|+|B|\right)^2 \geq |\mathbf{A}-\mathbf{B}|^2 = |A|^2+|B|^2-2|A||B|\cos\theta$$

Or

$$|A||B| \geq |A||B|\cos\theta = \vec{A}\cdot\vec{B}$$

When extended to a Hilbert space, the Schwartz inequality takes on the form above.

The Derivative of $|f(x)|^2$

A third useful relation comes from applying product rule of differentiation:

$$\frac{d}{dx}\left[|f(x)|^2\right] = \frac{d}{dx}\left[f(x)f(x)^*\right] = \dot{f}\,f^* + f\,\dot{f}^* = 2\operatorname{Re} f\,\dot{f}^*$$

APPENDIX J
THE UNCERTAINTY RELATION AND THE GAUSSIAN

The equality in the uncertainty relation actually holds for the Gaussian function. Demonstrating this fact requires some careful attention to detail.

We will use Gaskill's "Gaus" function.

$$Gaus(x) = e^{-\pi x^2}$$

Note that this is not the standard definition of a Gaussian distribution. The standard definition is

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-x^2}{2\sigma^2}}$$

The relation between the two is

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} Gaus\left(\frac{x}{\sqrt{2\pi\sigma}}\right)$$

The important Fourier relation is

$$F.T.\left[Gaus(x)\right] = Gaus(\xi)$$

Giving us the relation

$$F.T.\left[g_\sigma(x)\right] = \frac{1}{\sqrt{2\pi\sigma}} F.T.\left[Gaus\left(\frac{x}{\sqrt{2\pi\sigma}}\right)\right] = \frac{1}{\sqrt{2\pi\sigma}}\sqrt{2\pi\sigma}\,Gaus\left(\sqrt{2\pi\sigma}\,\xi\right) = Gaus\left(\sqrt{2\pi\sigma}\,\xi\right)$$

Let's look at the standard deviation of the Gaus function. Using a lower case subscript for a quantity in direct space and an upper case subscript for the quantity in Fourier space we know that:

$$\frac{1}{2\sigma_g^2} = \pi$$

and

$$\frac{1}{2\sigma_G^2} = \pi$$

or

$$\frac{1}{2\sigma_g^2} = 2\sigma_G^2$$

so that

$$\sigma_g = \frac{1}{\sqrt{2}}$$

Let's move on to normalization. Now we must take some care. The standard normalization of the gaussian is such that

$$\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1$$

For this normalization,

$$\int_{-\infty}^{\infty} x^2 g_\xi(x)\,dx = \sigma^2$$

which is the variance. We are not using this definition of the variance but rather, with *g* real:

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \frac{g_\sigma{}^2(x)}{\int_{-\infty}^{\infty} g_\sigma{}^2(x_0)\,dx_0}\,dx = \frac{\int_{-\infty}^{\infty} x^2 g_\sigma{}^2 dx}{\int_{-\infty}^{\infty} g_\sigma{}^2(x)\,dx}$$

The differences are surprisingly simple but subtle. First let's note that

$$g_\sigma{}^2(x) = \frac{1}{2\pi\sigma^2} e^{\frac{-x^2}{\sigma^2}}$$

We need to calculate two integrals. For the first we can use the gaussian normalization relation to obtain

$$\int_{-\infty}^{\infty} e^{\frac{-x^2}{\sigma^2}}\,dx = \sqrt{2\pi}\,\frac{\sigma}{\sqrt{2}} = \sqrt{\pi}\sigma$$

For the second we can use the gaussian rms relation to get

$$\int_{-\infty}^{\infty} x^2 e^{\frac{-x^2}{\sigma^2}}\,dx = \sqrt{\pi}\sigma\,\frac{\sigma^2}{2}$$

We finally have that

$$\langle x^2 \rangle = \frac{\sigma^2}{2}$$

the variance of the Fourier transform will be

$$\langle \xi^2 \rangle = \frac{\int_{-\infty}^{\infty} \xi^2 Gaus^2\left(\sqrt{2\pi}\sigma\xi\right) d\xi}{\int_{-\infty}^{\infty} Gaus^2\left(\sqrt{2\pi}\sigma\xi\right) d\xi} = \frac{\int_{-\infty}^{\infty} x^2 e^{-2\pi^2\sigma^2\xi^2} d\xi}{\int_{-\infty}^{\infty} e^{-2\pi^2\sigma^2\xi^2} d\xi}$$

At this point it is convenient to define

$$\sigma_\xi = \frac{1}{2\pi\sigma}$$

With this definition,

$$\langle \xi^2 \rangle = \frac{\int_{-\infty}^{\infty} x^2 e^{\frac{-\xi^2}{2\sigma_\xi^2}} d\xi}{\int_{-\infty}^{\infty} e^{\frac{-\xi^2}{2\sigma_\xi^2} 2\pi^2\sigma^2} d\xi} = \frac{\sigma_\xi^2}{2} = \frac{1}{2(2\pi\sigma)^2} = \frac{1}{8\pi^2\sigma^2}$$

And the corresponding uncertainty relation becomes

$$\langle x^2 \rangle \langle \xi^2 \rangle = \frac{\sigma^2}{2} \frac{1}{8\pi^2\sigma^2} = \frac{1}{16\pi^2}$$

or

$$\sqrt{\langle x^2 \rangle}\sqrt{\langle \xi^2 \rangle} = \frac{1}{4\pi}$$

which is the expected result.

APPENDIX K
THE UNCERTAINTY RELATION AND CGH'S

The uncertainty relation, and the calculations we have done using gaussian functions can be of great practical use when considering supports of CGH's. Although the results of this appendix will be used only after we discuss sampling, the material is introduced here because of its intimate connection to the derivations of the past two appendices.

Consider a one dimensional object, in direct space, with support of extent Δx and sample spacing δx . The roles of the numbers will be reversed in Fourier space where the extent will be $1/\delta x$ and the sample spacing will be $1/\Delta x$. With a constraint that the total number of sample points be ***N***, we have that $\Delta x = N\delta x$ and that $\frac{1}{\delta x} = N\frac{1}{\Delta x}$. Up to this point the constraints are naturally consistent. We now make an explicit assumption, i.e., that the grid is the same in both spaces so that $\Delta x = \frac{1}{\delta x}$. This assumption forces $\frac{1}{\delta x} = N\delta x$ or

$\delta x = \frac{1}{\sqrt{N}}$ and $\Delta x = \sqrt{N}$.

Let's consider the discrete analog of the Gaus function. Remembering that sigma is related to <u>half</u> the width of the function, the full fractional width is

$$\frac{2\sigma_g}{\Delta x} = \frac{2/\sqrt{2}}{\sqrt{N}} = \sqrt{\frac{2}{N}}$$

The full width is

$$\frac{2\sigma_g}{\Delta x} N = \sqrt{2N}$$

For example, in a field of 128 points, the RMS-width of the Gaus function in both domains will be 16.

Now we will take a short digression. A rule of thumb is that if a function in direct space is m points wide, its DFT is N/m, i.e., one m-th of the field. This rule would lead us to expect that the Gaus function should be $\sqrt{N}$ points wide. We could interpret a conflict into these two ways of looking at the width relations. The truth is that the two statement are looking at different kinds of widths. The important lesson to learn is that the geometrical scaling for "equal-sized" object and spectrum goes as the square-root of the space-bandwidth product, i.e., number of points.

APPENDIX L
FOURIER TRANSFORM TABLES

<u>Properties of one-dimensional Fourier transforms (see Gaskill, pg 199)</u>

Fourier transforms are useful because of their special relationship to linear shift-invariant systems. They are also useful because the Fourier exponentials represent plane waves, a basic element of wave optics. Much of the ease in working with Fourier transforms comes from solving problems by using known relations instead of calculating the transforms fresh every time. The following table collects together many of those relations.

Function	Fourier Transform
$f(x)$	$F(\xi)$
reflection $f(-x)$	$F(-\xi)$
conjugation $f^*(x)$	$F^*(-\xi)$
scaling $f\left(\frac{x}{b}\right)$	$\lvert b\rvert F(b\xi)$
shift $f(x \pm x_0)$	$e^{\pm 2\pi i x_0 \xi} F(\xi)$
multiplication by a linear phase factor $e^{\pm 2\pi i \xi_0 x} f(x)$	$F(\xi \mp \xi_0)$
Forward transform of a transform $F(x)$	$f(-\xi)$
kth order derivative $f^{(k)}(x) = \frac{d^k f}{dx^k}$	$(2\pi i \xi)^k F(\xi)$
kth order anti-derivative $f^{(-k)}(x)$	$(2\pi i \xi)^{-k} F(\xi)$
$h(x) f(x)$	$(H * F)(\xi)$
convolution $(h * f)(x)$	$H(\xi) F(\xi)$
self-convolution $(f * f)(x)$	$F^2(\xi)$
$h^*(x) f(x)$	Cross-correlation $(H^* \star F)(\xi)$
$\lvert f(x)\rvert^2$	Auto-correlation $(F^* \star F)(\xi)$

- Definitions

Convolution: $(h * f)(x) = \int_{-\infty}^{\infty} h(x - x_0) f(x_0) dx_0$

Correlation: $(h^* \star f)(x) = \int_{-\infty}^{\infty} h^*(x_0 - x) f(x_0) dx_0$

There are several variations of in the definition of the correlation. Some explicitly have a complex conjugate, e.g., $(h^* \star f)(x)$, some have the conjugate on the second rather than the first argument. Some use a plus sign in the integrand. Some use other symbols than the star. Some texts don't even use the correlation but prefer to consider it just another convolution. The definition we are using corresponds to the "complex correlation" in Bracewell's book.

- Relations:

At a discontinuity $f(x_0) = \frac{1}{2} \lim_{\varepsilon \to 0} \left[f(x_0 - \varepsilon) + f(x_0 + \varepsilon) \right]$

Parseval's Theorem: $\int_{-\infty}^{\infty} \lvert f(x)\rvert^2 dx = \int_{-\infty}^{\infty} \lvert F(\xi)\rvert^2 d\xi$

Plancheral's Theorem: $\int_{-\infty}^{\infty} h^*(x) f(x) dx = \int_{-\infty}^{\infty} H^*(\xi) F(\xi) d\xi$

<u>Common one-dimensional Fourier-transform pairs (Gaskill, pg. 201)</u>

The following are some common 1-D Fourier-transform pairs. Using the knowledge of such pairs, and avoiding constant recalculation is crucial to becoming comfortable with using Fourier transforms. You can get a good idea of the behavior of many optical systems, just by knowing a few of these relations.

Function	Fourier Transform
$f(x)$	$F(\xi)$
$\delta(x)$	1
1	$\delta(\xi)$
$\cos(2\pi\xi_0 x)$	$\frac{1}{2}\left[\delta(\xi-\xi_0)+\delta(\xi+\xi_0)\right]$
$\sin(2\pi\xi_0 x)$	$\frac{1}{2i}\left[\delta(\xi-\xi_0)-\delta(\xi+\xi_0)\right]$
$rect(x)$	$sinc(\xi)$
$sinc(x)$	$rect(\xi)$
$tri(x)$	$sinc^2(\xi)$
$\text{sgn}(x)$	$\frac{1}{i\pi\xi}$
$step(x)$	$\frac{1}{2}\delta(\xi)+\frac{1}{2\pi i\xi}$
$\frac{1}{i\pi x}$	$\text{sgn}(\xi)$
$e^{-\pi x^2}$	$e^{-\pi\xi^2}$
$e^{-\lvert x\rvert}$	$\frac{2}{1+(2\pi\xi)^2}$
$step(x)e^{-\lvert x\rvert}$	$\frac{1}{1+2\pi i\xi}$
$comb(x)$	$comb(\xi)$

- Definitions

$$rect(x)=\begin{cases}1 \text{ for } \lvert x\rvert<\frac{1}{2}\\ \frac{1}{2} \text{ for } \lvert x\rvert=\frac{1}{2}\\ 0 \text{ for } \lvert x\rvert>\frac{1}{2}\end{cases} \qquad sinc(x)=\frac{\sin(\pi x)}{\pi x} \qquad tri(x)=\begin{cases}1-\lvert x\rvert \text{ for } \lvert x\rvert<1\\ 0 \text{ for } \lvert x\rvert\ge 1\end{cases}$$

$$\text{sgn}(x)=\begin{cases}-1 \text{ for } x<0\\ 0 \text{ for } x=0\\ +1 \text{ for } x>0\end{cases} \qquad step(x)=\begin{cases}0 \text{ for } x<0\\ \frac{1}{2} \text{ for } x=0\\ 1 \text{ for } x>0\end{cases} \qquad comb(x)=\sum_{n=-\infty}^{\infty}\delta(x-n)$$

<u>Properties of two-dimensional Fourier transforms (Gaskill pg. 313, 327)</u>

The relations are in two dimensions are mostly straightforward extensions of those from one dimension. The exception has to do with the polar form. There is no angle of rotation in one dimension.

Function	Fourier Transform
$f(\pm x, \pm y)$	$F(\pm\xi, \pm\eta)$
$f^*(\pm x, \pm y)$	$F^*(\mp\xi, \mp\eta)$
$f(r, \theta+\theta_0)$	$F(\rho, \psi+\theta_0)$
$f\left(\frac{x}{x_0}, \frac{y}{y_0}\right)$	$\lvert x_0 y_0\rvert F(x_0\xi, y_0\eta)$
$f\left(\frac{r}{r_0}\right) = f\left[\sqrt{\left(\frac{x}{r_0}\right)^2 + \left(\frac{y}{r_0}\right)^2}\right]$	$\lvert r_0^2\rvert F(r_0\rho)$
$f(x \pm x_0, y \pm y_0)$	$e^{\pm 2\pi i(x_0\xi + y_0\eta)} F(\xi, \eta)$
$e^{\pm 2\pi i(\xi_0 x + \eta_0 y)} f(x, y)$	$F(\xi \mp \xi_0, \eta \mp \eta_0)$
Forward transform of a transform $F(x, y)$	$f(-\xi, -\eta)$
$f^{(j,k)}(x, y) = \frac{\partial^{j+k} f}{\partial x^j \partial y^k}$	$(2\pi i\xi)^{(j)} \cdot (2\pi i\eta)^{(k)} F(\xi, \eta)$
$h(x, y) f(x, y)$	$(H{*}{*}F)(\xi, \eta)$
$(h{*}{*}f)(x, y)$	$H(\xi, \eta) F(\xi, \eta)$
$(f{*}{*}f)(x, y)$	$F^2(\xi, \eta)$
$h^*(x, y) f(x, y)$	$(H^* \star\star F)(\xi, \eta)$
$\lvert f(x, y)\rvert^2$	$(F^* \star\star F)(\xi, \eta)$

- Definitions

Convolution: $(h{*}{*}f)(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x - x_0, y - y_0) f(x_0, y_0) dx_0 dy_0$

Correlation: $(h^* \star\star f)(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(x_0 - x, y_0 - y) f(x_0, y_0) dx_0 dy_0$

- Relations

Parseval's Theorem: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lvert f(x, y)\rvert^2 dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lvert F(\xi, \eta)\rvert^2 d\xi d\eta$

Plancheral's Theorem: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(x, y) f(x, y) dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H^*(\xi, \eta) F(\xi, \eta) d\xi d\eta$

<u>Common two-dimensional Fourier transform pairs (Gaskill, pg. 317)</u>

As with 1-D Fourier transforms, there are many useful relations for 2-D transforms. The following table contains a few of them.

$f(x,y)$	$F(\xi,\eta)$
$\delta(x)\delta(y)$	1
1	$\delta(\xi)\delta(\eta)$
$\cos[2\pi(\xi_0 x+\eta_o y)]$	$\frac{1}{2}[\delta(\xi-\xi_0)\delta(\eta-\eta_0)+\delta(\xi+\xi_0)\delta(\eta+\eta_0)]$
$\sin[2\pi(\xi_0 x+\eta_o y)]$	$\frac{1}{2i}[\delta(\xi-\xi_0)\delta(\eta-\eta_0)-\delta(\xi+\xi_0)\delta(\eta+\eta_0)]$
$rect(x)rect(y)$	$sinc(\xi)sinc(\eta)$
$cyl(r)$	$\frac{\pi}{4}somb(\rho)$
$somb(r)$	$\frac{4}{\pi}cyl(\rho)$
$sinc(x)sinc(y)$	$rect(\xi)rect(\eta)$
$tri(x)tri(y)$	$sinc^2(\xi)sinc^2(\eta)$
$comb(x)comb(y)$	$comb(\xi)comb(\eta)$

- Definitions

$$cyl(r)=\begin{cases}1 & \text{for } 0\le r<\tfrac{1}{2}\\ \tfrac{1}{2} & \text{for } r=\tfrac{1}{2}\\ 0 & \text{for } r>\tfrac{1}{2}\end{cases}$$ the cylinder function, $$somb(\rho)=\frac{2J_1(\pi\rho)}{\pi\rho}$$ the sombrero function

$J_1(r)$ Bessel function of the first kind (*J*) and first order