Module 05 Lectures 17-20
Mathematics for Optics (Optical Sciences 503)
Chapter Two: Functions of a Complex Variable

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Lecture 17 --

- The Cauchy principal value for one family of integral

We now move to the case handled in M&W, i.e., an analytic function $f(z)$ where we wish to find the Cauchy principal value of the integral

$$P\int_a^b \frac{f(x)}{x-x_0}dx$$

We can expand $f(z)$ using the its Taylor series into

$$f(z)=\sum_{n=0}^{\infty} f_n(z-x_0)^n = f_0 + f_1(z-x_0) + \ldots$$

We define the function

$$g(z)=\frac{f(z)}{z-x_0}=\frac{f_0}{z-x_0}+f_1+f_2(z-x_0)+\ldots$$

Looking back at its Laurent series expansion

$$g(z)=\frac{a_{-1}}{z-x_0}+a_0+a_1(z-x_0)+\ldots$$

it is clear that $f_0 = a_{-1}$.

We use the results of our discussion from Lecture #16 to conclude that

$$P\int_a^b \frac{f(x)}{x-x_0}dx = P\int_a^b g(x)dx = i\pi a_{-1}$$

When the pole is very slightly above or below the real-axis, we can use our results to conclude that

$$\int_a^b \frac{f(x)}{x-x_0 \mp i\varepsilon}dx = P\int_a^b \frac{f(x)}{x-x_0}dx \pm i\pi a_{-1} = P\int_a^b \frac{f(x)}{x-x_0}dx \pm i\pi f(x_0)$$

The difference in sign comes from how we close the contour. If we wish to have the pole outside the contour, as we have assumed, then there are two possibilities. If the pole is below the real-axis, we close in the upper half-plane. If, on the other hand, the pole is slightly above the real-axis, then we close in the lower half-plane. Closing in the lower half-plane forces us to transverse the small semi-circle in the opposite, negative, direction. The last term comes from the relation

$$f(x_0) = \lim_{z\to x_0}\left[f(z)\right] = \lim_{z\to x_0}\left[(z-x_0)g(z)\right] = a_{-1}$$

The next relation shown in M&W is

$$\frac{1}{x-x_0 \mp i\varepsilon} = P\frac{1}{x-x_0} \pm i\pi\delta(x-x_0)$$

The two relations that are used to get there are the sifting property of the Dirac delta function

$$\int_a^b f(x)\delta(x-x_0)dx = \int_a^b f(x)\delta(x_0-x)dx = f(x_0)$$

and the definition

$$\int_a^b P\frac{1}{x-x_0}dx = P\int_a^b \frac{1}{x-x_0}dx$$

Integrating over both sides of

$$\frac{1}{x-x_0 \mp i\varepsilon} = P\frac{1}{x-x_0} \pm i\pi\delta(x-x_0)$$

gets us back to

$$\int_a^b \frac{f(x)}{x-x_0 \mp i\varepsilon}dx = P\int_a^b \frac{f(x)}{x-x_0}dx \pm i\pi f(x_0)$$

- Finding Residues (M&W page 66)

The structure behind the formula for finding residues is to remove the singularity in question from the function. Once the singularity is removed, the resulting function can be expanded in a Taylor series. We then use the formula for finding coefficients of a Taylor series.

Suppose that the function we are dealing with is $f(z)$ and that its Laurent series about the point z_0 is

$$f(z) = \sum_{-M}^{\infty} a_m (z-z_0)^m$$

We see immediately that we are dealing with an M^{th} order pole. Now we proceed to remove the singularity through multiplication by $(z-z_0)^M$, so that

$$(z-z_0)^M f(z) = (z-z_0)^M \sum_{m=-M}^{\infty} a_m (z-z_0)^m = \sum_{m=-M}^{\infty} a_m (z-z_0)^{m+M}$$

We now change indices by substituting $n = m + M$ or $m = n - M$, giving us

$$(z-z_0)^M f(z) = \sum_{m=0}^{\infty} a_{n-M} (z-z_0)^n$$

We are after a_{-1} so $n - M = -1$, or, $n = M - 1$. We can use the formula that we usually apply to calculating Taylor series coefficients. It now becomes that formula for calculating residues.

$$a_{-1} = \frac{1}{(M-1)!} \left\{ \frac{d^{M-1}}{dz^{M-1}} \left[(z-z_0)^M f(z) \right] \right\}_{z=z_0}$$

If the singularity is essential, i.e. $M = \infty$, then the residue is found directly from the Laurent series expansion.

- Showing that the integral on the large semicircle goes to zero

M&W has an example on page 66, we are going to examine only a portion of that example. We look at the contour integral $\int_C \frac{dz}{1+z^2}$. In fact, we are interested only in the segment of C that is the large semi-circle centered at the origin.

$$\int_{\substack{\text{Large}\\\text{Semi-Circle}}} \frac{dz}{1+z^2} = \lim_{r\to\infty} \int_0^{\pi} \frac{ire^{i\theta} d\theta}{1+r^2 e^{2i\theta}} \approx \lim_{r\to\infty} \frac{ir}{r^2} \int_0^{\pi} e^{-i\theta} d\theta$$

$$= \lim_{r\to\infty} \frac{i}{r} \int_0^{\pi} i \sin\theta \, d\theta = -\left[-\cos\theta\right]_0^{\pi} \lim_{r\to\infty} \frac{1}{r}$$

$$= (-1-1) \lim_{r\to\infty} \frac{1}{r} = -2 \lim_{r\to\infty} \frac{1}{r} = 0$$

Lecture 18 --

- The Identity Theorem

Two functions, regular in a region R, will be equal everywhere in the region if any one of the following conditions is fulfilled:

1. They have the same values for all points within some sub-region.
2. They have the same values for all points on an arc within the region.
3. They have the same values on a denumerably (countably) infinite number of points with a limit point that lies within the region.

The identity theorem is useful in extending the definition of a function from the real axis to the complex plane. In M&W, the example given is the exponential, that on the real axis has the Taylor expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

In the complex plane,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

This function reduces to e^x on the real axis. The identity theorem guarantees us that we have arrived at the only extension into the complex plane. Why? Suppose that there was another function $f(z)$ that reduced to e^x on the real axis, then the identity theorem tells us that

$$f(z) = e^z$$

Let's look at two common trigonometric functions. We begin with

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Its continuation to the complex plane is

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Of particular interest is its value on the imaginary axis where $z = iy$

$$\cos(iy) = \frac{e^{-y} + e^{+y}}{2} = \frac{e^{+y} + e^{-y}}{2} = \cosh(y)$$

This extension is the relation of hyperbolic cosine to the cosine function.

Moving to

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

we have

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

on the imaginary axis

$$\sin(iy) = \frac{e^{-y} - e^{+y}}{2i} = -i\frac{e^{+y} - e^{-y}}{2} = -i\sinh(y)$$

The hyperbolic sine is related to the sine.

- The Schwartz Reflection Principle

If a function $f(z)$ is analytic in a region that contains a finite portion of the real axis, and $f(z)$ is real on the real axis, then $f(z^*) = [f(z)]^*$ throughout the region.

- Analytic Continuation

An analytic function is represented by its power series within its region of convergence. This region of convergence extends to the nearest singularity. Refer to the figure.

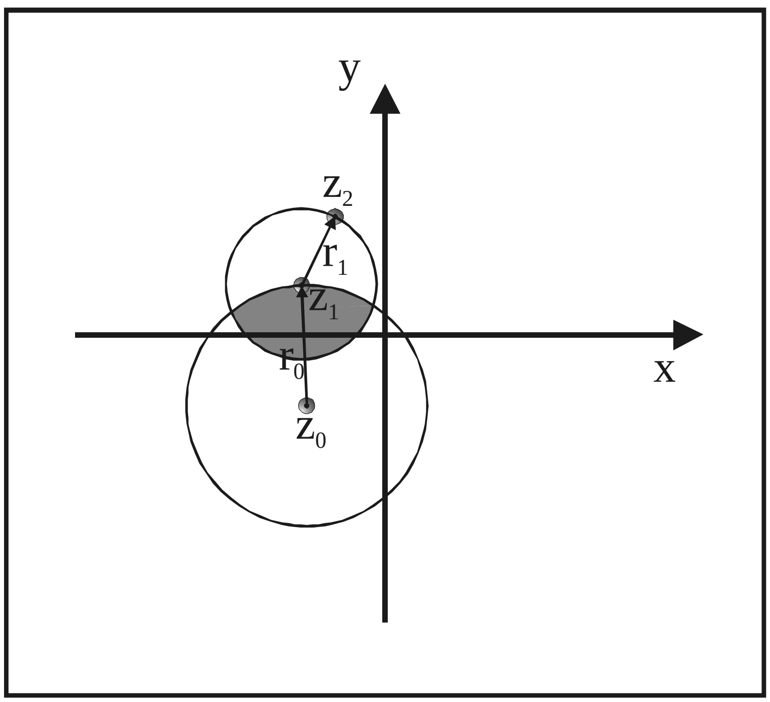

Figure 1: Convergence regions

The figure illustrates a function $f_0(z)$ with a singularity. We expand the function that is analytic about the point z_0 within its radius of convergence r_0. Next we consider a function $f_1(z)$ that is analytic about the point z_1 within its radius of convergence r_1. The two functions overlap in a region of finite extent. Applying the identity theorem to the region of overlap, we can extend the function from the first region to cover the second region. This process is known as analytic continuation. Using this procedure we can cover the entire complex plane with the exception of the singularities.

Analytic continuation has been applied to the recovery of images that are truncated. For example, an imaging system delivers an image, but we have a CCD (charge coupled device) sensor that acquires the image only over its surface. We would like to look slightly beyond the edge of the sensor. Or, a region of pixels on the CCD may be dead. We would like to interpolate the image into that region. To a small extent, this method works. Unfortunately, noise and sampling (the finite spacing of sensor elements) strongly limits the effectiveness of such efforts.

Lecture 19 --

- Dispersion Relations

The term dispersion relates to the dependence on frequency, or equivalently on wavelength, of optical interactions. Dispersion is often used to simply mean the function of the index of refraction versus frequency $n(\nu)$. In the present context, it has a slightly different meaning: the relation between the real and the imaginary parts of the index of refraction.

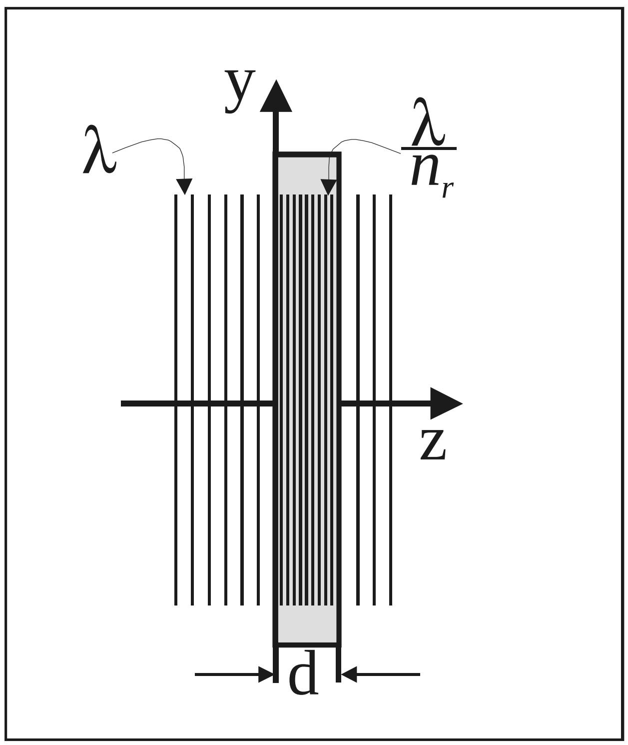

Figure 2: Wavelength and Index

The model we examine is a plane wave incident normally on a thin, but not zero-thickness, planar element with uniform index of refraction. See Figure 2. The model is actually widely applicable, because of the fact that at small distance scales, most waves can be approximated by plane waves, and most objects can be considered as a stack of thin planar layers.

Let's consider one component of a monochromatic electric field: $E_x(x,y,z,t) = u(x,y,z,t)$. The quantity u is called the complex amplitude. We will be considering only the vicinity of the z-axis, so we abbreviate the notation to $E_x(0,0,z,t)=u(z,t)$. We have chosen ϕ_0 to be zero. The notation $z=0^{\pm}$ means $z=0\pm\varepsilon$ with ε being the usual small number.

Just upstream of the element,

$$u(z=0^-,t)=e^{i(-\omega t+\phi_0)}=e^{-i\omega t}$$

We will now use two different interpretations of the effect of the wave passing through the element. First, we use the temporal interpretation that a phase of the wave enters the element at time t_0 and exits at time t_1. The complex-amplitude transmission of the element

$$\tau(\omega)=a(\omega)e^{i\omega(t_1-t_0)}$$

We will be considering the angular frequency ω to be a complex variable $\omega=\omega_r+i\omega_i$. We will be operating in the ω complex plane.

$$\tau(\omega)=a(\omega)e^{i\omega_r(t_1-t_0)}e^{-\omega_i(t_1-t_0)}$$

Causality tells us that $t_1-t_0>0$. The exponential attenuation in the upper half-plane, becomes an exponential gain in the lower half-plane. The integral along the large semi-circle will diverge in the lower-half-plane, therefore, we will be closing contours in the upper half plane. The imaginary angular frequency plays the important role of dampening the wave as the element thickness increases.

Now we move to a spatial interpretation of the transmittance. Just downstream of the element,

$$u(0^+,t)=\tau(\omega)e^{i(kd-\omega t)}$$

where $\tau(\omega)$ is the same complex amplitude transmittance as we just considered. This transmittance can be expressed as

$$\tau(\omega)=e^{i(k_1-k_0)d}=e^{i\left(\frac{2\pi}{\lambda_0/n(\omega)}-\frac{2\pi}{\lambda_0}\right)d}=e^{2\pi i\frac{d}{\lambda_0}[n(\omega)-1]}=e^{ik_0d[n(\omega)-1]}$$

In order to include attenuation, we consider the index of refraction to be complex

$$n(\omega)=n_r(\omega)+i\,n_i(\omega)$$

giving an exponential attenuation in the complex amplitude transmittance

$$\tau(\omega)=e^{ik_0[n_r(\omega)-1+in_i(\omega)]}=e^{ik_0[n_r(\omega)-1]}e^{-k_0\,n_i(\omega)}$$

The complex index of refraction is a classical physical quantity that is arguably smooth enough to be analytic. We now examine the index itself. We begin with the Cauchy integral theorem, using the variables as illustrated in the following figure.

$$n(\omega)=\frac{1}{2\pi i}\int_C \frac{n(\omega_0)}{\omega-\omega_0}d\omega_0$$

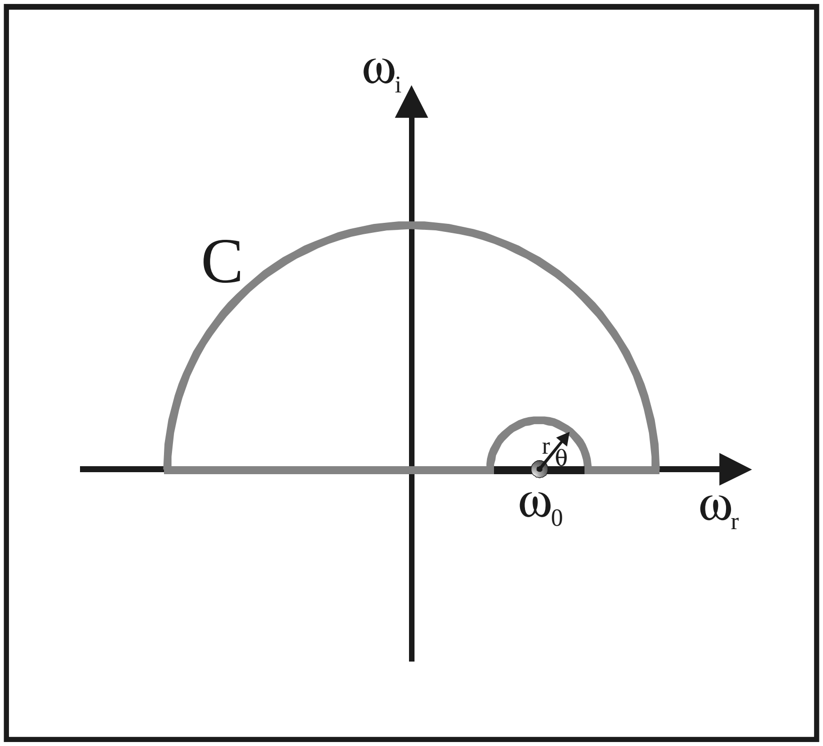

Figure 3: The Omega-Plane

Closing in the upper half-plane eliminates the diverging contribution to the transmittance from the large semi-circle. We use the results from Lecture 17 to write

$$n(\omega)=\frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{n(\omega_0)}{\omega-\omega_0}d\omega_0+\frac{1}{2}n(\omega)$$

or

$$n(\omega)=\frac{1}{\pi i}P\int_{-\infty}^{\infty}\frac{n(\omega_0)}{\omega-\omega_0}d\omega_0$$

Breaking both sides into their real and imaginary parts we have

$$n_r(\omega)+in_i(\omega)=\frac{1}{\pi i}P\int_{-\infty}^{\infty}\frac{n_r(\omega_0)+in_i(\omega_0)}{\omega-\omega_0}d\omega_0$$

Keeping careful track of the *i*'s and limiting the integration over ω to the real axis, we have

$$n_r(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{n_i(\omega_0)}{\omega - \omega_0} d\omega_0$$

$$n_i(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{-n_r(\omega_0)}{\omega - \omega_0} d\omega_0$$

We get a slightly different form with different assumptions as we see in the M&W equations 5-13 and 5-14 where the angular frequencies are taken to be only positive and the index is symmetric.

- Example. Another interesting relation comes from the Cauchy-Riemann equations. The index is analytic so that

$$\frac{\partial n_r}{\partial \omega_r} = \frac{\partial n_i}{\partial \omega_i} \quad \text{and} \quad \frac{\partial n_i}{\partial \omega_r} = -\frac{\partial n_r}{\partial \omega_i}$$

As an example, suppose we wished to use a material for an optical pulse compressor, and wish to know how the absorptive part of the index might influence its behavior. The index we desire is quadratic, i.e., $n_r = a\omega_r^2 + f_0(\omega_i)$, so that $\frac{\partial n_r}{\partial \omega_r} = 2a\omega_r$. The first Cauchy-Riemann equation tells us that $\frac{\partial n_i}{\partial \omega_i} = 2a\omega_r$. Integrating gives us $n_i = 2a\omega_r\omega_i + f_1(\omega_r)$. Employing the second Cauchy-Riemann equation gives

$$\frac{\partial n_i}{\partial \omega_r} = 2a\omega_i + \frac{\partial f_1(\omega_r)}{\partial \omega_r} = -\frac{\partial f_0(\omega_i)}{\partial \omega_i} = -\frac{\partial n_r}{\partial \omega_i}$$

This equation will be satisfied if $\frac{\partial f_1(\omega_r)}{\partial \omega_r} = 0$ and $\frac{\partial f_0(\omega_i)}{\partial \omega_i} = -2a\omega_i$

Integrating yields $f_1(\omega_r) = 0$ and $f_0(\omega_i) = -a\omega_i^2$. So that

$$n_r = a\left(\omega_r^2 - \omega_i^2\right) \quad \text{and} \quad n_i = 2a\omega_r\omega_i$$

The complex amplitude transmittance of our small slab will be

$$\tau(\omega) = e^{ia\left(\omega_r^2 - \omega_i^2\right)} e^{-2a\omega_r\omega_i}$$

Interestingly enough, the absorptive part will not only cause absorption, but will also shift the frequency response.

Lecture 20 --

- Curve Tangents

A convenient way to describe a curve is to use a parameter that gives the coordinates in terms of a position on the curve, A commonly used parameter is arc-length commonly designated by the variable *s*. In the *z*-plane,

$$z = z(s) = x(s) + iy(s)$$

In the *w*-plane,

$$w(s) = u(s) + iv(s)$$

The variable *s* represents arc-length in the *z*-plane, but not in the *w*-plane. It does however couple corresponding points:

$$w(s) = w[z(s)]$$

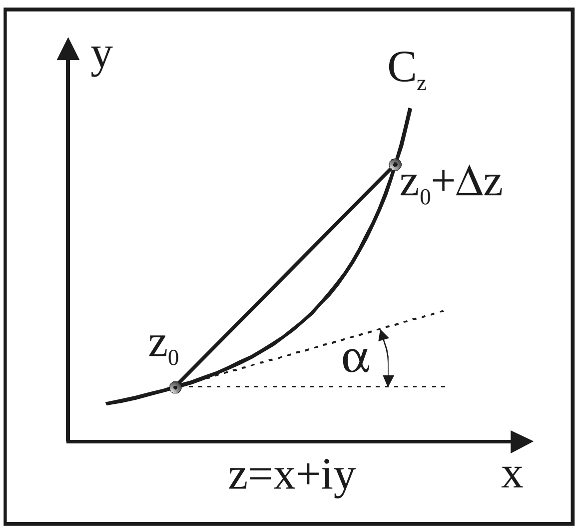

Figure 4: Curve in the z-plane

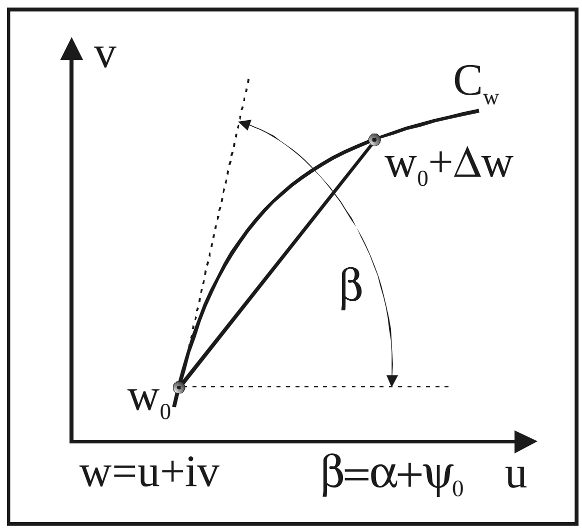

Figure 5: Curve in the w-plane

We wish to examine the tangents to the curves in the figures. In the *z*-plane,

$$\Delta z = \Delta x + i\Delta y \approx \frac{dx}{ds}\Delta s + i\frac{dy}{ds}\Delta s = \left(\frac{dx}{ds} + i\frac{dy}{ds}\right)\Delta s$$

In the limit as $\Delta s \to 0$, Δz is tangent to the cure in the *z*-plane. In the *w*-plane,

$$\Delta w = \Delta u + i\Delta v \approx \left(\frac{du}{ds} + i\frac{dv}{ds}\right)\Delta s$$

approaches the tangent to the curve there.

In the limit, we can use the chain rule to get

$$\frac{dw}{ds} = \left(\frac{dw}{dz}\right)\frac{dz}{ds}$$

Let's write the quantities of interest in the complex polar form:

$$\frac{dw}{ds} = ae^{i\beta}, \quad \frac{dz}{ds} = a_z e^{i\alpha}, \quad \frac{dw}{dz} = a_w e^{i\psi_0}$$

The results from the chain rule then translate to

$$\frac{dw}{ds} = ae^{i\beta} = \frac{dw}{dz}\frac{dz}{ds} = a_w e^{i\psi_0} a_z e^{i\alpha} = a_z a_w e^{i(\alpha+\psi_0)}$$

We see that

$$\beta = \alpha + \psi_0$$

Of course, if $\left.\frac{dw}{ds}\right|_{s_0} = 0$, then the relation does not hold. In that case, we say that $z(s_0)$ is a critical point.

- Conformal Mapping

A mapping of curves from one plane to another, where intersection angles are preserved, is said to me conformal. See the figures

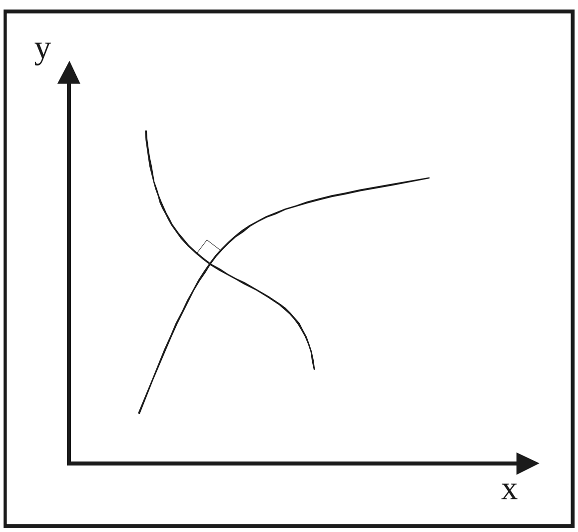

Figure 6: Two curves intersecting in the z-plane

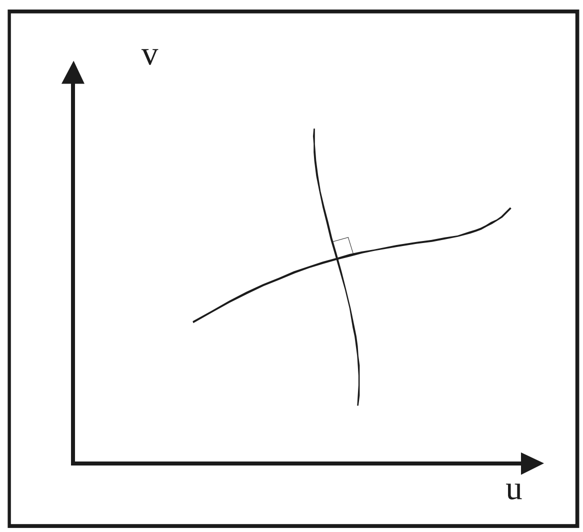

Figure 7: Two curves intersecting in the w-plane

For two curves intersecting at the point z_0, the relations between the angles in the *z*-plane an the *w*-plane are

$$\beta_1 = \alpha_1 + \psi_0 \text{ and } \beta_2 = \alpha_2 + \psi_0$$

Subtracting the two equations reveals that

$$\beta_2 - \beta_1 = \alpha_2 - \alpha_1$$

For analytic functions, the mappings are conformal. The exceptions are at singular points and critical points.

A particular case is for orthogonal curves, i.e., curves intersecting at right angles. The conformal mapping takes orthogonal curves into orthogonal curves. This make them good for coordinate transformations.

- Example: $w = z^2$

  $$w = u + iv = x^2 - y^2 + 2ixy$$

  Equating the real and imaginary parts separately we have

  $$u = x^2 - y^2$$

  and

  $$v = 2xy$$

  If we consider the line $y = x$, then, $w = x^2 - x^2 + 2ix^2 = 0 + i2x^2$ and $u = 0$, $v = 2x^2$.

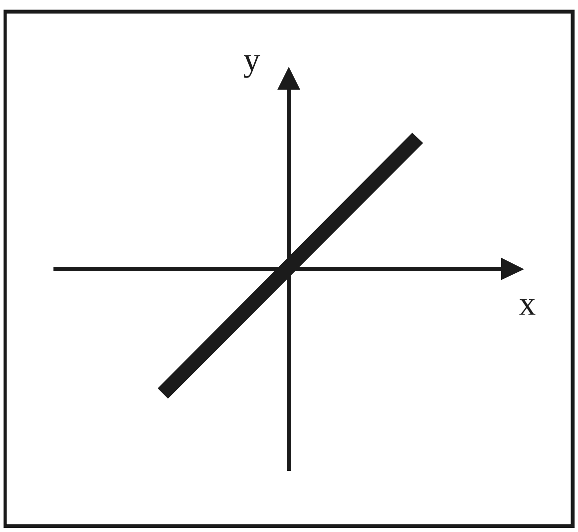

Figure 8: Line in the z-plane

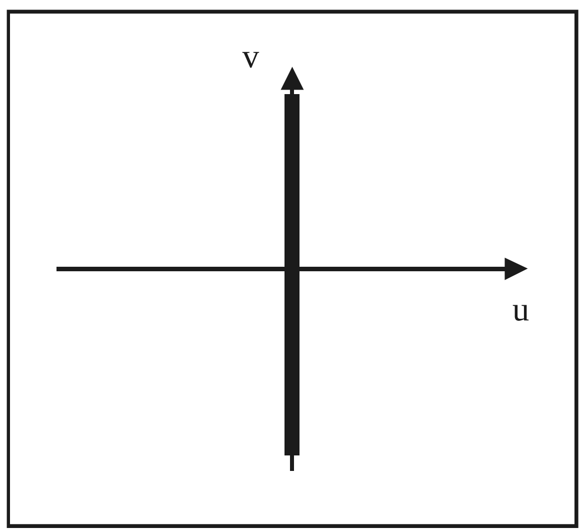

Figure 9:Line in the w-plane

The line, $x = 1$, maps into

$$w = 1 - y^2 + 2iy$$

Separating the real and imaginary parts gives,

$$u = 1 - y^2$$

$$v = 2y$$

Or,

$$u = 1 - \frac{v^2}{4}$$

a parabola.

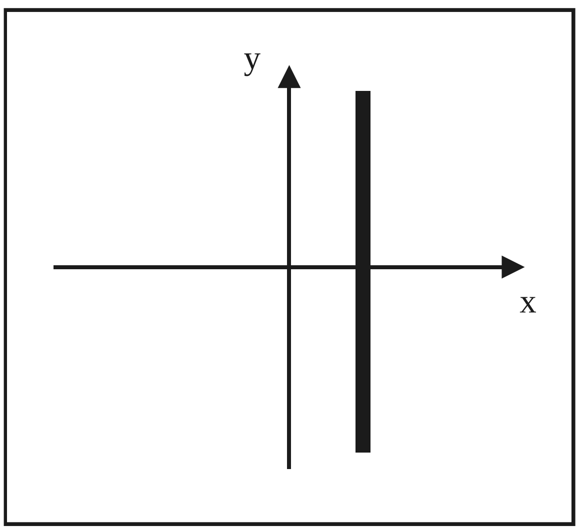

Figure 10: Line in the z-plane

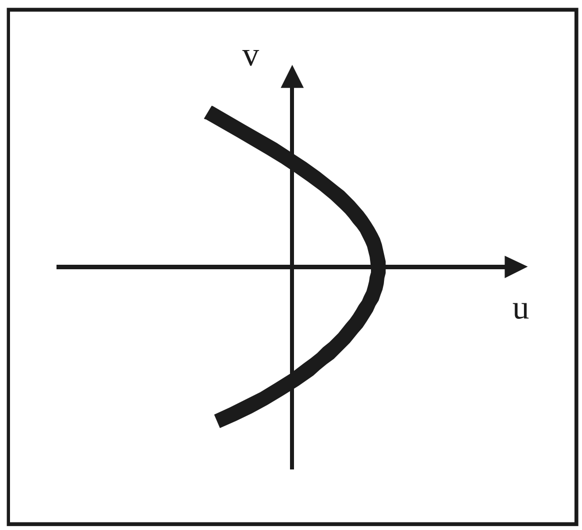

Figure 11: Parabola in the w-plane

- Derivative of an analytic function's inverse

If $w = f(z)$ is analytic at and $\left.\frac{df}{dz}\right|_{z=z_0} \neq 0$, then there exists a neighborhood of $w(z_0)$ in the w-plane in which $w = f(z)$ has a unique inverse, $z = F(w)$. The derivatives at z_0 are related by

$$\frac{dF}{dw} = \frac{1}{df/dz}$$

- An application of conformal mapping: bent and tapered light guides

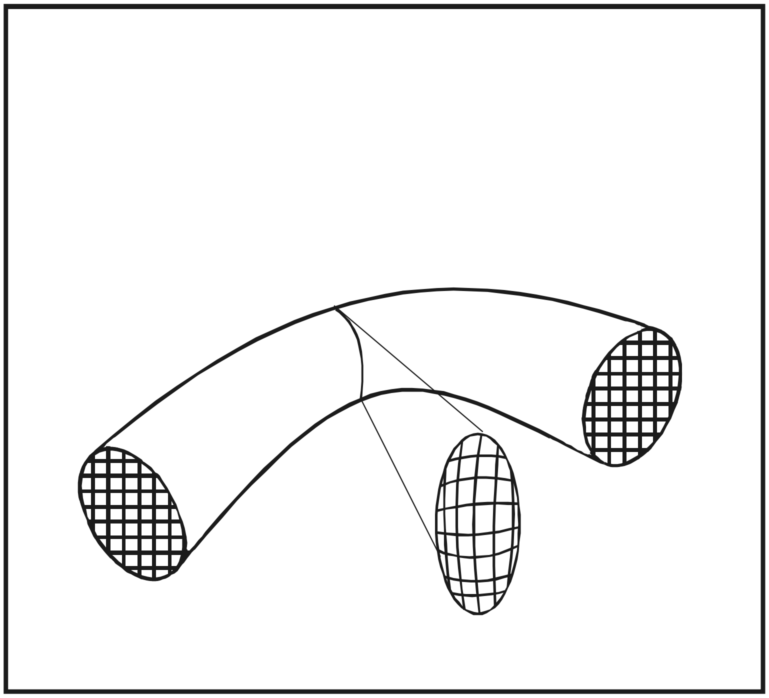

Figure 12: Distortion at the bend

- Solving Laplace's equation, the quadrupole magnet as a charged particle lens.

  The optical analog of the quadrupole magnet is a stack of two cylindrical lenses, one positive, one negative, oriented at right angles.

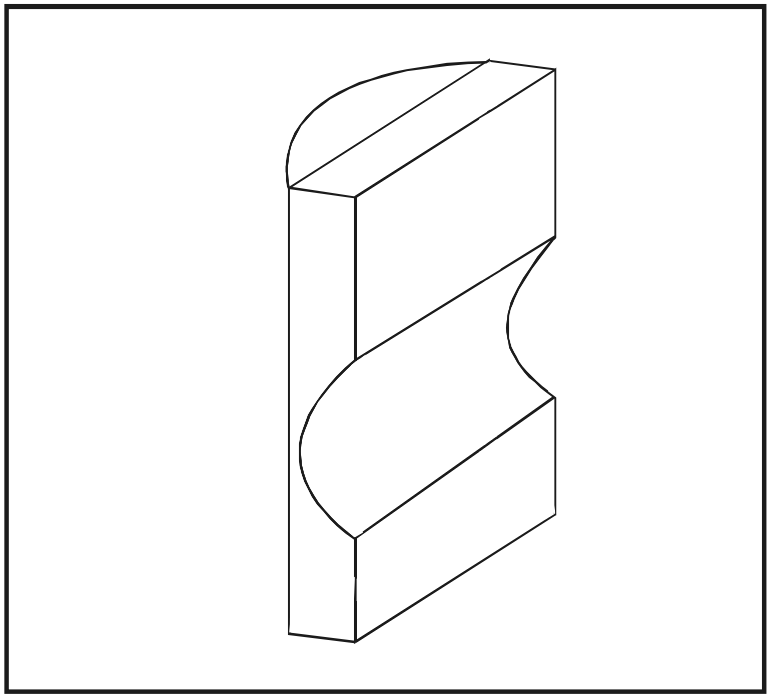

Figure 6: Optical analog of a quadruplole beam-focusing lens